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The problems in this set are similar to the ones that students applying to universities and colleges usually solve at their entrance examinations. We recommend them to everyone who is applying. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given system of equations over the real numbers.

$$\begin{aligned} 2^{\frac{x-y}{2}} + 2^{\frac{x-y}{4}} &= 2 \\ 3^{\log(2y-x)} &= 1 \end{aligned}$$

2. The perimeter of triangle ABC is 45 cm, and

$$\frac{\sin \alpha}{3} = \frac{\sin \beta}{5} = \frac{\sin \gamma}{7}$$

Find the sides of the triangle.

3. The intersection points of the circle $x^2 + y^2 - 6x - 2y - 15 = 0$ and the coordinate axes form a quadrilateral. What percent of the area of the circle is the area of this quadrilateral?
4. From a point of a straight highway, a straight side road branches off to the right at an angle of 65° . It leads to village A , which is 7.5 km away. Going 3 km further on the highway, another straight side road branches off to the left at an angle of 45° , and it leads to village B , which is 5.5 km away. How far apart are the two villages (as the crow flies)?
5. Solve the given equation over the real numbers.

$$\sin^4 x + \sin^4 \left(x + \frac{\pi}{4} \right) + \sin^4 \left(x - \frac{\pi}{4} \right) = \frac{5}{4}$$

6. Consider three cones K_1 , K_2 , and K_3 . Their radii r_1 , r_2 , r_3 are three consecutive terms of an arithmetic sequence, and their heights m_1 , m_2 , m_3 are three consecutive terms of an arithmetic sequence with the same common difference. Prove that their volumes V_1 , V_2 , V_3 cannot be three consecutive terms of an arithmetic sequence.
7. We have a rectangular metal sheet with sides a and $2a$ dm. We cut out congruent squares at the four corners, and we fold up the remaining rectangles on the sides to make a box that is open at the top. How long should the sides of the cut-out squares be so that the volume of the box is as large as possible? What is this maximal volume? What should the value of a be if the maximal volume has to be 3 dm^3 ?
8. The sum of 1981 different natural numbers is 3 924 359. Prove that at least two of these numbers are even.