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The problems in this set are similar to the ones that students applying to universities and colleges usually solve at their entrance examinations. We recommend them to everyone who is applying. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given system of equations over the real numbers.

$$\begin{aligned} 2^{\frac{x-y}{2}} + 2^{\frac{x-y}{4}} &= 2 \\ 3^{\log(2y-x)} &= 1 \end{aligned}$$

Solution: Let $t = 2^{\frac{x-y}{4}}$. Then $t > 0$ and $2^{\frac{x-y}{2}} = t^2$, so the first equation becomes

$$\begin{aligned} t^2 + t - 2 &= 0 \\ (t+2)(t-1) &= 0 \end{aligned}$$

Since $t > 0$, $t = 1$, and so $\frac{x-y}{4} = 0$, that is, $x = y$. The second equation says that $\log(2y-x) = 0$, that is, $2y-x = 1$. With $x = y$ this gives $y = 1$. So the only solution is $x = 1$, $y = 1$. (Notice that $2y-x = 1 > 0$, so the logarithm is defined.)

2. The perimeter of triangle ABC is 45 cm, and

$$\frac{\sin \alpha}{3} = \frac{\sin \beta}{5} = \frac{\sin \gamma}{7}$$

Find the sides of the triangle.

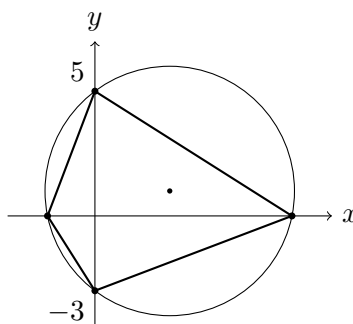
Solution: By the Law of Sines, the sides are proportional to the sines of the opposite angles: $a : b : c = \sin \alpha : \sin \beta : \sin \gamma = 3 : 5 : 7$. So $a = 3k$, $b = 5k$, and $c = 7k$ for some $k > 0$, and

$$3k + 5k + 7k = 15k = 45 \quad \text{so} \quad k = 3$$

The sides are 9 cm, 15 cm, and 21 cm. (Such a triangle exists since $9 + 15 > 21$; its largest angle is 120° .)

3. The intersection points of the circle $x^2 + y^2 - 6x - 2y - 15 = 0$ and the coordinate axes form a quadrilateral. What percent of the area of the circle is the area of this quadrilateral?

Solution: We complete the squares: $(x-3)^2 + (y-1)^2 = 25$. So the circle has center $(3, 1)$ and radius 5, and its area is 25π .



With $y = 0$: $x^2 - 6x - 15 = 0$, so $x = 3 \pm 2\sqrt{6}$. With $x = 0$: $y^2 - 2y - 15 = 0$, so $y = 5$ or $y = -3$. The quadrilateral has its diagonals on the coordinate axes, so its diagonals are perpendicular, with lengths $4\sqrt{6}$ and 8. Its area is

$$\frac{4\sqrt{6} \cdot 8}{2} = 16\sqrt{6} \approx 39.19$$

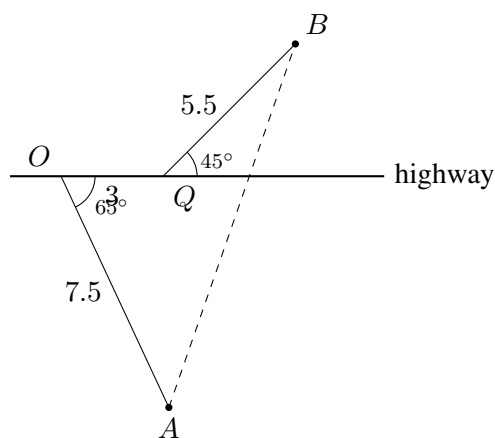
and the ratio of the two areas is

$$\frac{16\sqrt{6}}{25\pi} \approx 0.499$$

So the area of the quadrilateral is about 49.9% of the area of the circle.

4. From a point on a straight highway, a straight side road branches off to the right at an angle of 65° . It leads to village A , which is 7.5 km away. Going 3 km further on the highway, another straight side road branches off to the left at an angle of 45° , and it leads to village B , which is 5.5 km away. How far apart are the two villages (as the crow flies)?

Solution: We measure both angles from the direction in which we travel on the highway. We place the highway on the x -axis, with the first junction at the origin and the direction of travel to the right. The second junction is then at $(3, 0)$.



The road to the right makes an angle of -65° with the positive x -axis, and the road to the left makes an angle of 45° . So

$$\begin{aligned} A &= (7.5 \cos 65^\circ, -7.5 \sin 65^\circ) \approx (3.170, -6.797) \\ B &= (3 + 5.5 \cos 45^\circ, 5.5 \sin 45^\circ) \approx (6.889, 3.889) \end{aligned}$$

and the distance between the villages is

$$AB \approx \sqrt{(6.889 - 3.170)^2 + (3.889 + 6.797)^2} = \sqrt{3.719^2 + 10.686^2} \approx 11.3 \text{ km}$$

5. Solve the given equation over the real numbers.

$$\sin^4 x + \sin^4 \left(x + \frac{\pi}{4}\right) + \sin^4 \left(x - \frac{\pi}{4}\right) = \frac{5}{4}$$

Solution: We use the identity $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$ for each term.

$$\begin{aligned}\sin^2 x &= \frac{1 - \cos 2x}{2} \\ \sin^2 \left(x + \frac{\pi}{4}\right) &= \frac{1 - \cos \left(2x + \frac{\pi}{2}\right)}{2} = \frac{1 + \sin 2x}{2} \\ \sin^2 \left(x - \frac{\pi}{4}\right) &= \frac{1 - \cos \left(2x - \frac{\pi}{2}\right)}{2} = \frac{1 - \sin 2x}{2}\end{aligned}$$

Let $c = \cos 2x$ and $s = \sin 2x$. Squaring each line and adding them, the equation becomes

$$\begin{aligned}\frac{(1 - c)^2 + (1 + s)^2 + (1 - s)^2}{4} &= \frac{5}{4} \quad \text{multiply by 4} \\ 1 - 2c + c^2 + 2 + 2s^2 &= 5 \quad s^2 = 1 - c^2 \\ 3 - 2c + c^2 + 2 - 2c^2 &= 5 \\ -c^2 - 2c &= 0 \\ c(c + 2) &= 0\end{aligned}$$

Since $c = \cos 2x$ cannot be -2 , we have $\cos 2x = 0$, so $2x = \frac{\pi}{2} + k\pi$, that is,

$$x = \frac{\pi}{4} + k \cdot \frac{\pi}{2} \quad \text{where } k \text{ is any integer.}$$

Check with $x = \frac{\pi}{4}$: $\sin^4 \frac{\pi}{4} + \sin^4 \frac{\pi}{2} + \sin^4 0 = \frac{1}{4} + 1 + 0 = \frac{5}{4}$.

6. Consider three cones K_1 , K_2 , and K_3 . Their radii r_1 , r_2 , r_3 are three consecutive terms of an arithmetic sequence, and their heights m_1 , m_2 , m_3 are three consecutive terms of an arithmetic sequence with the same common difference. Prove that their volumes V_1 , V_2 , V_3 cannot be three consecutive terms of an arithmetic sequence.

Solution: Let d be the common difference, $r = r_2$, and $m = m_2$. Then $r_1 = r - d$, $r_3 = r + d$, $m_1 = m - d$, and $m_3 = m + d$. We need to assume that $d \neq 0$: if $d = 0$, then the three cones are the same, and their volumes do form an arithmetic sequence (with common difference 0).

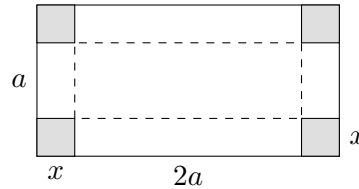
Three numbers V_1 , V_2 , V_3 are consecutive terms of an arithmetic sequence exactly when $V_1 + V_3 = 2V_2$. Using $V = \frac{\pi}{3}r^2m$,

$$\begin{aligned}\frac{3}{\pi}(V_1 + V_3 - 2V_2) &= (r - d)^2(m - d) + (r + d)^2(m + d) - 2r^2m \\ &= (r^2m - r^2d - 2rdm + 2rd^2 + d^2m - d^3) \\ &\quad + (r^2m + r^2d + 2rdm + 2rd^2 + d^2m + d^3) - 2r^2m \\ &= 4rd^2 + 2md^2 = 2d^2(2r + m)\end{aligned}$$

Since $r > 0$, $m > 0$, and $d \neq 0$, this is positive. So $V_1 + V_3 > 2V_2$, and the volumes cannot be three consecutive terms of an arithmetic sequence.

7. We have a rectangular metal sheet with sides a and $2a$ dm. We cut out congruent squares at the four corners, and we fold up the remaining rectangles on the sides to make a box that is open at the top. How long should the sides of the cut-out squares be so that the volume of the box is as large as possible? What is this maximal volume? What should the value of a be if the maximal volume has to be 3 dm^3 ?

Solution: Let x denote the side of the cut-out squares, where $0 < x < \frac{a}{2}$.



The box has a base of $(2a - 2x)$ by $(a - 2x)$, and height x . So its volume is

$$V(x) = x(a - 2x)(2a - 2x) = 4x^3 - 6ax^2 + 2a^2x$$

We look for the critical numbers:

$$\begin{aligned} V'(x) &= 12x^2 - 12ax + 2a^2 = 0 && \text{divide by 2} \\ 6x^2 - 6ax + a^2 &= 0 \\ x &= \frac{6a \pm \sqrt{36a^2 - 24a^2}}{12} = \frac{(3 \pm \sqrt{3})a}{6} \end{aligned}$$

Only $x = \frac{(3 - \sqrt{3})a}{6} \approx 0.211a$ is less than $\frac{a}{2}$. Here $V''(x) = 24x - 12a = -4\sqrt{3}a < 0$, so this is a maximum. For this x ,

$$a - 2x = \frac{\sqrt{3}}{3}a, \quad 2a - 2x = \frac{3 + \sqrt{3}}{3}a$$

and so the maximal volume is

$$V = \frac{(3 - \sqrt{3})a}{6} \cdot \frac{\sqrt{3}a}{3} \cdot \frac{(3 + \sqrt{3})a}{3} = \frac{(9 - 3)\sqrt{3}}{54}a^3 = \frac{\sqrt{3}}{9}a^3 \approx 0.192a^3 \text{ dm}^3$$

If this volume is 3 dm^3 , then

$$\frac{\sqrt{3}}{9}a^3 = 3 \quad \text{so} \quad a^3 = \frac{27}{\sqrt{3}} = 9\sqrt{3} \quad \text{and} \quad a = \sqrt[3]{9\sqrt{3}} = 3^{\frac{5}{6}} \approx 2.50 \text{ dm}$$

8. The sum of 1981 different natural numbers is 3 924 359. Prove that at least two of these numbers are even.

Solution: Suppose that at most one of the numbers is even.

If none of them is even, then the 1981 numbers are different odd numbers. The sum of 1981 different odd numbers is at least the sum of the 1981 smallest odd numbers:

$$1 + 3 + 5 + \dots + (2 \cdot 1981 - 1) = 1981^2 = 3\,924\,361$$

But this is more than 3 924 359, so this is impossible.

If exactly one of them is even, then the other 1980 numbers are odd. The sum of an even number of odd numbers is even, and adding one more even number, the total is even. But 3 924 359 is odd, so this is impossible too.

So at least two of the numbers are even.