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This time, instead of a regular problem set, we collected problems of one type for practice: *equations and inequalities with a parameter*. In each problem, a denotes a real parameter.

1. Solve the given equation for x , and discuss the solutions for every value of a .

$$(a^2 - 1)x = 2a^2 + 3a + 1$$

Solution: We factor both sides: $(a - 1)(a + 1)x = (2a + 1)(a + 1)$.

If $a \neq \pm 1$, we can divide by $a^2 - 1$, and there is exactly one solution:

$$x = \frac{(2a + 1)(a + 1)}{(a - 1)(a + 1)} = \frac{2a + 1}{a - 1}$$

If $a = 1$, the equation becomes $0 \cdot x = 6$, and there is no solution.

If $a = -1$, the equation becomes $0 \cdot x = 0$, and every real number is a solution.

2. Solve the given system of equations.

$$\begin{aligned} 4x + ay &= 9 \\ 2ax + 18y &= -27 \end{aligned}$$

Solution: We eliminate x : multiply the first equation by a and the second one by 2, and subtract.

$$\begin{aligned} 4ax + a^2y &= 9a \\ 4ax + 36y &= -54 && \text{subtract} \\ (a^2 - 36)y &= 9a + 54 \\ (a - 6)(a + 6)y &= 9(a + 6) \end{aligned}$$

If $a \neq \pm 6$, then $y = \frac{9}{a - 6}$, and from the first equation,

$$4x = 9 - \frac{9a}{a - 6} = \frac{-54}{a - 6} \quad \text{so} \quad x = \frac{27}{2(6 - a)}$$

So there is exactly one solution: $x = \frac{27}{2(6 - a)}$, $y = \frac{9}{a - 6}$.

If $a = 6$, the equations are $4x + 6y = 9$ and $12x + 18y = -27$, that is, $4x + 6y = -9$. These contradict each other, so there is no solution.

If $a = -6$, the equations are $4x - 6y = 9$ and $-12x + 18y = -27$, and the second one is -3 times the first one. So there are infinitely many solutions: x can be any real number, and $y = \frac{4x - 9}{6}$.

3. Solve the given equation.

$$\sqrt{x-a} + \sqrt{x} = 1$$

Solution: For the square roots, we need $x \geq 0$ and $x \geq a$. Also, $\sqrt{x-a} = 1 - \sqrt{x}$ must be non-negative, so $\sqrt{x} \leq 1$.

$$\begin{aligned}\sqrt{x-a} &= 1 - \sqrt{x} && \text{square both sides} \\ x-a &= 1 - 2\sqrt{x} + x \\ 2\sqrt{x} &= 1+a \\ \sqrt{x} &= \frac{1+a}{2}\end{aligned}$$

This requires $0 \leq \frac{1+a}{2} \leq 1$, that is, $-1 \leq a \leq 1$. Then $x = \frac{(1+a)^2}{4}$, and $x-a = \frac{(1-a)^2}{4} \geq 0$, so x is in the domain. Checking: $\sqrt{x-a} + \sqrt{x} = \frac{1-a}{2} + \frac{1+a}{2} = 1$.

So if $-1 \leq a \leq 1$, the only solution is $x = \frac{(1+a)^2}{4}$, and for any other value of a , there is no solution.

4. Find all values of a for which exactly one root of the equation

$$(a^2 + 1)x^2 + (a+1)x - 2 = 0$$

is in the interval $(0, 1)$.

Solution: Let $f(x) = (a^2 + 1)x^2 + (a+1)x - 2$. Since $a^2 + 1 > 0$, the graph of f is a parabola opening upward. Since $f(0) = -2 < 0$, the equation always has two roots, one negative and one positive. The negative root is never in $(0, 1)$, so we need the positive root to be less than 1. This happens exactly when $f(1) > 0$:

$$f(1) = a^2 + 1 + a + 1 - 2 = a^2 + a = a(a+1) > 0$$

So exactly one root is in $(0, 1)$ if and only if $a < -1$ or $a > 0$.

5. Find all values of a for which the value of the polynomial

$$(a^2 - 1)x^2 + 2(a-1)x + 2$$

is positive for every x .

Solution: We consider three cases.

If $a = 1$, the polynomial is the constant 2, which is always positive.

If $a = -1$, the polynomial is $-4x + 2$, which is not positive for every x .

If $a \neq \pm 1$, the polynomial is quadratic. It is positive for every x exactly when its leading coefficient is positive and its discriminant is negative:

$$a^2 - 1 > 0 \quad \text{and} \quad \frac{D}{4} = (a-1)^2 - 2(a^2 - 1) = (a-1)(a-1-2a-2) = -(a-1)(a+3) < 0$$

The first condition gives $a < -1$ or $a > 1$, and the second one gives $a < -3$ or $a > 1$. Both hold when $a < -3$ or $a > 1$.

So the polynomial is positive for every x if and only if $a < -3$ or $a \geq 1$.

6. Solve the given equation.

$$\frac{2}{a} - \frac{2^x + a}{a \cdot 2^x} = \frac{2^x}{2^x - a}$$

Solution: We need $a \neq 0$ and $2^x \neq a$. Let $t = 2^x$, so $t > 0$ and $t \neq a$. The left-hand side is

$$\frac{2}{a} - \frac{t + a}{at} = \frac{2t - t - a}{at} = \frac{t - a}{at}$$

and so the equation becomes

$$\frac{t - a}{at} = \frac{t}{t - a} \quad \text{or} \quad (t - a)^2 = at^2$$

The left-hand side is positive (since $t \neq a$) and $t^2 > 0$, so there is no solution if $a < 0$. If $a > 0$, we take square roots:

$$t - a = \pm \sqrt{a}t \quad \text{so} \quad t = \frac{a}{1 + \sqrt{a}} \quad \text{or} \quad t = \frac{a}{1 - \sqrt{a}}$$

The first value is always positive, and it is not equal to a . The second value exists and is positive only if $\sqrt{a} < 1$, that is, $0 < a < 1$; it is not equal to a either. So:

- if $a \leq 0$, there is no solution;
- if $0 < a < 1$, there are two solutions: $x = \log_2 \frac{a}{1 + \sqrt{a}}$ and $x = \log_2 \frac{a}{1 - \sqrt{a}}$;
- if $a \geq 1$, there is one solution: $x = \log_2 \frac{a}{1 + \sqrt{a}}$. (For example, if $a = 1$, then $x = -1$.)

7. Solve the given equation.

$$\frac{\log x}{\log \left(x - \frac{a^2 - 1}{4} \right)} = 2$$

Solution: Let $c = \frac{a^2 - 1}{4}$. We need $x > 0$, $x - c > 0$, and $x - c \neq 1$ (so that the denominator is not zero). Then

$$\begin{aligned} \log x &= 2 \log (x - c) = \log (x - c)^2 \\ x &= (x - c)^2 \\ 0 &= x^2 - (2c + 1)x + c^2 \end{aligned}$$

The discriminant is $(2c + 1)^2 - 4c^2 = 4c + 1 = a^2$, and $2c + 1 = \frac{a^2 + 1}{2}$, so

$$x = \frac{\frac{a^2 + 1}{2} \pm a}{2} = \frac{(a \pm 1)^2}{4}$$

Now we check the conditions. For $x_1 = \frac{(a + 1)^2}{4}$, we get $x_1 - c = \frac{1 + a}{2}$, which must be positive and not 1: $a > -1$ and $a \neq 1$. For $x_2 = \frac{(a - 1)^2}{4}$, we get $x_2 - c = \frac{1 - a}{2}$, so we need $a < 1$ and $a \neq -1$. (If $x - c > 0$, then $x = (x - c)^2 > 0$ automatically.) So:

- if $a < -1$, the only solution is $x = \frac{(a - 1)^2}{4}$;
- if $a = -1$ or $a = 1$, there is no solution;

- if $-1 < a < 1$, the solutions are $x = \frac{(a+1)^2}{4}$ and $x = \frac{(a-1)^2}{4}$ (if $a = 0$, these are the same: $x = \frac{1}{4}$);
- if $a > 1$, the only solution is $x = \frac{(a+1)^2}{4}$.

8. Consider the equation $\cos x \cdot \cot x - \sin x = a \cos 2x$. For a given value of a , how many solutions does this equation have with $0 \leq x \leq 2\pi$?

Solution: For $\cot x$, we need $\sin x \neq 0$, so $x \neq 0, \pi, 2\pi$. We simplify the left-hand side:

$$\cos x \cdot \frac{\cos x}{\sin x} - \sin x = \frac{\cos^2 x - \sin^2 x}{\sin x} = \frac{\cos 2x}{\sin x}$$

So the equation is

$$\frac{\cos 2x}{\sin x} = a \cos 2x \quad \text{or} \quad \cos 2x \left(\frac{1}{\sin x} - a \right) = 0$$

So either $\cos 2x = 0$ or $\sin x = \frac{1}{a}$.

The equation $\cos 2x = 0$ has four solutions in $[0, 2\pi]$: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, for every value of a .

The equation $\sin x = \frac{1}{a}$ has solutions only if $a \neq 0$ and $\left| \frac{1}{a} \right| \leq 1$, that is, $|a| \geq 1$. If $|a| = 1$, it has one solution ($\frac{\pi}{2}$ or $\frac{3\pi}{2}$); if $|a| > 1$, it has two. These new solutions are different from the four above, except when $\frac{1}{a} = \pm \frac{\sqrt{2}}{2}$, that is, $a = \pm\sqrt{2}$: then they are two of the four. In summary, the number of solutions is

- 4 if $|a| < 1$ or $|a| = \sqrt{2}$,
- 5 if $|a| = 1$,
- 6 if $|a| > 1$ and $|a| \neq \sqrt{2}$.