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The problems in this set are similar to the ones that students applying to universities and colleges usually solve at their entrance examinations. We recommend them to everyone who is applying. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Simplify the expression $x + x^3 + x^5 + \dots + x^{27}$.

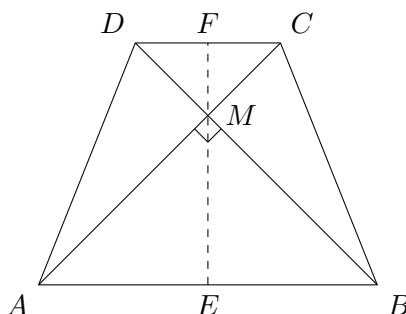
Solution: This is the sum of a geometric sequence with first term x and common ratio x^2 . The exponents are $1, 3, 5, \dots, 27$, so there are $\frac{27-1}{2} + 1 = 14$ terms. If $x^2 \neq 1$, then

$$x + x^3 + x^5 + \dots + x^{27} = x \cdot \frac{(x^2)^{14} - 1}{x^2 - 1} = \frac{x(x^{28} - 1)}{x^2 - 1}$$

If $x = 1$, the sum is 14, and if $x = -1$, the sum is -14 .

2. The height of an isosceles trapezoid is 10 units. Find its area if its diagonals are perpendicular to each other.

Solution: Let $ABCD$ be the trapezoid with parallel sides AB and CD , and let M be the intersection of the diagonals.



Because the trapezoid is isosceles, it is symmetric about the line EF through the midpoints E of AB and F of CD , and M is on this line. So $MA = MB$, and since the diagonals are perpendicular, triangle ABM is an isosceles right triangle. Its height ME is half of its hypotenuse: $ME = \frac{AB}{2}$. The same way, $MF = \frac{CD}{2}$. So the height of the trapezoid is

$$10 = EF = ME + MF = \frac{AB + CD}{2}$$

and the area is

$$A = \frac{AB + CD}{2} \cdot 10 = 10 \cdot 10 = 100 \text{ unit}^2$$

3. A farm raises rabbits. At the beginning of three consecutive years, the farm buys 1000, 3000, and 3000 rabbits. At the end of the third year, there are 8261 rabbits on the farm. What is the average yearly percentage of growth?

Solution: Suppose that every year the number of rabbits is multiplied by q . The 1000 rabbits bought at the beginning of the first year grow for three years, the first 3000 for two years, and the second 3000 for one year. So

$$\begin{aligned} 1000q^3 + 3000q^2 + 3000q &= 8261 && \text{add 1000} \\ 1000(q^3 + 3q^2 + 3q + 1) &= 9261 \\ 1000(q + 1)^3 &= 9261 \\ (q + 1)^3 &= 9.261 \\ q + 1 &= 2.1 \\ q &= 1.1 \end{aligned}$$

So the number of rabbits grows by 10% every year.

4. Two vertices of an isosceles triangle are $(1, 6)$ and $(-5, 2)$. Where can the third vertex be?

Solution: Let $P(1, 6)$ and $Q(-5, 2)$. Then $PQ = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$. There are three cases.

Case 1. PQ is the base. Then the third vertex is on the perpendicular bisector of PQ . The midpoint of PQ is $(-2, 4)$ and the slope of PQ is $\frac{4}{6} = \frac{2}{3}$, so the perpendicular bisector is

$$y - 4 = -\frac{3}{2}(x + 2) \quad \text{or} \quad 3x + 2y = 2$$

The third vertex can be any point of this line except the midpoint $(-2, 4)$.

Case 2. PQ is a leg, and the apex is P . Then the third vertex is on the circle with center P and radius PQ :

$$(x - 1)^2 + (y - 6)^2 = 52$$

except the two points of this circle that are on line PQ : Q itself and $(7, 10)$.

Case 3. PQ is a leg, and the apex is Q . Then the third vertex is on the circle

$$(x + 5)^2 + (y - 2)^2 = 52$$

except P and $(-11, -2)$.

5. Solve the given equation.

$$x^2\sqrt{\sin 2x} + 4 = 4\sqrt{\sin 2x} + x^2$$

Solution: For the square roots, we need $\sin 2x \geq 0$. We rearrange and factor.

$$\begin{aligned} x^2\sqrt{\sin 2x} + 4 &= 4\sqrt{\sin 2x} + x^2 \\ x^2\sqrt{\sin 2x} - x^2 - 4\sqrt{\sin 2x} + 4 &= 0 \\ x^2(\sqrt{\sin 2x} - 1) - 4(\sqrt{\sin 2x} - 1) &= 0 \\ (x^2 - 4)(\sqrt{\sin 2x} - 1) &= 0 \end{aligned}$$

If $x^2 - 4 = 0$, then $x = 2$ or $x = -2$. But $\sin 4 < 0$ (since $\pi < 4 < 2\pi$), so $x = 2$ is not in the domain, while $\sin(-4) = -\sin 4 > 0$, so $x = -2$ is a solution.

If $\sqrt{\sin 2x} = 1$, then $\sin 2x = 1$, and so $2x = \frac{\pi}{2} + 2k\pi$, that is, $x = \frac{\pi}{4} + k\pi$.

So the solutions are $x = -2$ and $x = \frac{\pi}{4} + k\pi$, where k is any integer.

6. Sketch the set of points in the coordinate plane whose coordinates satisfy the given equation.

$$\frac{y - x^4}{\left(\log \frac{9 - y}{5}\right) \sqrt{4x^2 - 4xy + y^2 - 9}} = 0$$

Solution: A fraction is zero when its numerator is zero and its denominator is defined and not zero. So we need:

- $y = x^4$ (the numerator is zero),
- $\frac{9 - y}{5} > 0$, that is, $y < 9$ (the logarithm is defined),
- $\frac{9 - y}{5} \neq 1$, that is, $y \neq 4$ (the logarithm is not zero),
- $4x^2 - 4xy + y^2 - 9 > 0$ (the square root is defined and not zero).

The last condition is $(2x - y)^2 > 9$, that is, $|2x - y| > 3$. On the curve $y = x^4$, the conditions become

$$-\sqrt{3} < x < \sqrt{3}, \quad x \neq \pm\sqrt{2}, \quad \text{and} \quad |2x - x^4| > 3$$

The expression $2x - x^4$ is never greater than 3: its maximum is at $x = \sqrt[3]{\frac{1}{2}} \approx 0.79$, where it is less than 1.2. So we need $2x - x^4 < -3$, that is, $x^4 - 2x - 3 > 0$. Since $x = -1$ is a root, we can factor:

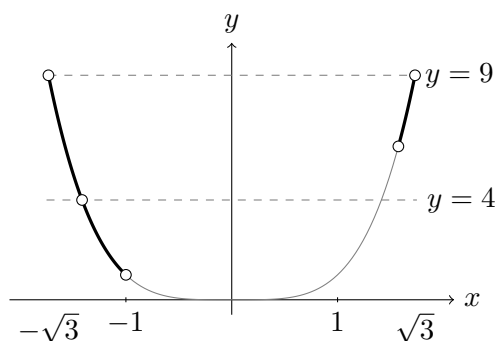
$$x^4 - 2x - 3 = (x + 1)(x^3 - x^2 + x - 3)$$

The cubic factor is increasing (its derivative $3x^2 - 2x + 1$ is always positive), and it has one real root, $r \approx 1.575$. So $x^4 - 2x - 3 > 0$ exactly when $x < -1$ or $x > r$.

Putting everything together, the set consists of the points of the curve $y = x^4$ for which

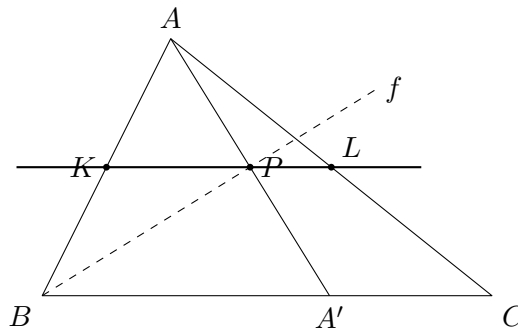
$$-\sqrt{3} < x < -1 \text{ but } x \neq -\sqrt{2}, \quad \text{or} \quad r < x < \sqrt{3}$$

These are two arcs of the curve, with the point $(-\sqrt{2}, 4)$ removed from the first one. The endpoints are not included.



7. From one vertex of a triangle, we draw the perpendiculars to the interior and exterior angle bisectors of the other two vertices. Prove that the feet of these perpendiculars lie on a line.

Solution: Let the triangle be ABC , and let us draw the perpendiculars from A . Let f be the interior angle bisector at B , and let P be the foot of the perpendicular from A to f . Let A' be the reflection of A about f .



The reflection about f takes the ray BA to the ray BC , because f bisects the angle between them. So A' is on line BC . Since f is the perpendicular bisector of AA' , the point P is the midpoint of AA' .

Now apply the dilation with center A and ratio $\frac{1}{2}$. It takes A' to P , and it takes line BC to the line KL , where K and L are the midpoints of AB and AC (the midline of the triangle that is parallel to BC). So P is on the midline KL .

The same argument works for the exterior angle bisector at B : the reflection about the exterior bisector takes line BA to line BC as well (ray BA goes to the ray opposite to ray BC). So the reflection of A is again on line BC , and the foot of the perpendicular is again on KL . And the same holds for the two angle bisectors at C .

So all four feet of the perpendiculars lie on the midline KL .

8. Find the minimum value of the expression $y - 2x$ if

$$x^2 + 2y^2 = 8.$$

Solution: Let $k = y - 2x$. We need the values of k for which the line $y = 2x + k$ and the ellipse $x^2 + 2y^2 = 8$ have a common point.

$$\begin{aligned} x^2 + 2(2x + k)^2 &= 8 \\ x^2 + 8x^2 + 8kx + 2k^2 - 8 &= 0 \\ 9x^2 + 8kx + 2k^2 - 8 &= 0 \end{aligned}$$

This quadratic equation has a real solution if and only if its discriminant is non-negative.

$$\begin{aligned} (8k)^2 - 4 \cdot 9(2k^2 - 8) &\geq 0 \\ 64k^2 - 72k^2 + 288 &\geq 0 \\ 288 &\geq 8k^2 \\ 36 &\geq k^2 \end{aligned}$$

So $-6 \leq k \leq 6$, and the minimum value of $y - 2x$ is -6 . It is attained when the discriminant is zero: $x = -\frac{8k}{18} = \frac{8}{3}$

and $y = 2x + k = -\frac{2}{3}$. Check: $\left(\frac{8}{3}\right)^2 + 2\left(-\frac{2}{3}\right)^2 = \frac{64}{9} + \frac{8}{9} = 8$, and $-\frac{2}{3} - \frac{16}{3} = -6$.