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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Is it true that the given number is rational?

$$\frac{\sqrt{3} \cdot \sqrt{6}}{3\sqrt{2} - \sqrt{32} + \sqrt{8}}$$

Solution: We simplify the radicals: $\sqrt{32} = 4\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$.

$$\frac{\sqrt{3} \cdot \sqrt{6}}{3\sqrt{2} - \sqrt{32} + \sqrt{8}} = \frac{\sqrt{18}}{3\sqrt{2} - 4\sqrt{2} + 2\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2}} = 3$$

So yes, the number is rational: it is 3.

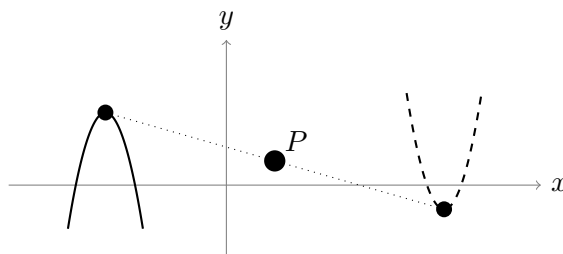
2. Find the equation of the image of the parabola $y = -2x^2 - 20x - 47$ under the reflection about the point $P(2, 1)$.

Solution: The reflection about $P(2, 1)$ takes the point (x, y) to $(4 - x, 2 - y)$, since P is the midpoint of the two points. So a point (X, Y) is on the image if and only if $(4 - X, 2 - Y)$ is on the original parabola:

$$\begin{aligned} 2 - Y &= -2(4 - X)^2 - 20(4 - X) - 47 \\ Y &= 2(16 - 8X + X^2) + 80 - 20X + 49 \\ Y &= 2X^2 - 36X + 161 \end{aligned}$$

So the equation of the image is $y = 2x^2 - 36x + 161$.

Check with the vertices: the original parabola is $y = -2(x + 5)^2 + 3$, with vertex $(-5, 3)$, and the image is $y = 2(x - 9)^2 - 1$, with vertex $(9, -1)$. The midpoint of these two vertices is indeed $(2, 1)$.



3. Let P be any point inside the base AB of the isosceles triangle ABC . Prove that the circles circumscribed around triangles ACP and BCP have equal radii.

Solution: Let R_1 and R_2 be the radii of the circles circumscribed around triangles ACP and BCP . By the Law of Sines, in triangle ACP the side CP is opposite the angle at A , and in triangle BCP the side CP is opposite the angle at B . So

$$2R_1 = \frac{CP}{\sin \angle A} \quad \text{and} \quad 2R_2 = \frac{CP}{\sin \angle B}$$

The triangle is isosceles with base AB , so $\angle A = \angle B$, and therefore $R_1 = R_2$.

4. Which real numbers x satisfy the given equation? (a is a real parameter.)

$$\frac{1+a^2}{\sin x} = 4a \cos x$$

Solution: We need $\sin x \neq 0$. Multiplying both sides by $\sin x$,

$$1+a^2 = 4a \sin x \cos x = 2a \sin 2x$$

If $a = 0$, this says $1 = 0$, so there is no solution. If $a \neq 0$, then $\sin 2x = \frac{1+a^2}{2a}$. But

$$\left| \frac{1+a^2}{2a} \right| \geq 1, \quad \text{since} \quad 1+a^2-2|a| = (|a|-1)^2 \geq 0$$

with equality only if $|a| = 1$. So there are solutions only if $a = \pm 1$.

If $a = 1$: $\sin 2x = 1$, so $2x = \frac{\pi}{2} + 2k\pi$ and $x = \frac{\pi}{4} + k\pi$.

If $a = -1$: $\sin 2x = -1$, so $2x = -\frac{\pi}{2} + 2k\pi$ and $x = -\frac{\pi}{4} + k\pi$.

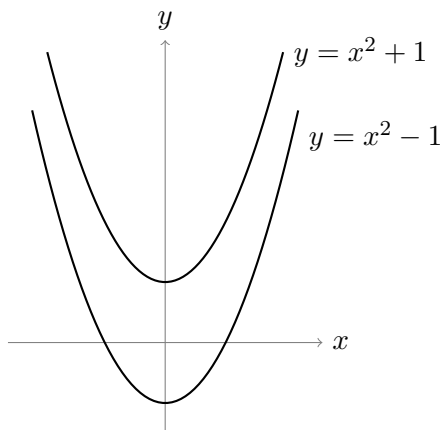
In both cases $\sin x \neq 0$, so these are solutions (k is any integer). For any other value of a , there is no solution.

5. Which points of the plane have coordinates that satisfy the equation $x^4 - 2x^2y + y^2 - 1 = 0$?

Solution: The first three terms form a perfect square:

$$x^4 - 2x^2y + y^2 - 1 = (y - x^2)^2 - 1 = 0$$

So $y - x^2 = 1$ or $y - x^2 = -1$. The points are the points of the two parabolas $y = x^2 + 1$ and $y = x^2 - 1$.



6. Find all values of m for which the roots of the equation $x^4 - (3m + 2)x^2 + m^2 = 0$ form an arithmetic sequence.

Solution: With $u = x^2$, the equation is $u^2 - (3m + 2)u + m^2 = 0$. For four real roots, we need two different positive values of u . If these are p^2 and q^2 with $0 < p < q$, then the roots are $-q, -p, p, q$. These form an arithmetic sequence if $q - p = p - (-p) = 2p$, that is, $q = 3p$. So the two values of u are p^2 and $9p^2$. By Vieta's formulas,

$$p^2 + 9p^2 = 10p^2 = 3m + 2 \quad \text{and} \quad p^2 \cdot 9p^2 = 9p^4 = m^2$$

From the second equation, $m = 3p^2$ or $m = -3p^2$.

If $m = 3p^2$: $10p^2 = 9p^2 + 2$, so $p^2 = 2$ and $m = 6$. Check: $x^4 - 20x^2 + 36 = 0$ gives $x^2 = 2$ or 18 , and the roots $-3\sqrt{2}, -\sqrt{2}, \sqrt{2}, 3\sqrt{2}$ form an arithmetic sequence.

If $m = -3p^2$: $10p^2 = -9p^2 + 2$, so $p^2 = \frac{2}{19}$ and $m = -\frac{6}{19}$. Check: the equation is $x^4 - \frac{20}{19}x^2 + \frac{36}{361} = 0$, so $x^2 = \frac{2}{19}$ or $\frac{18}{19}$, and again the larger root is three times the smaller positive root.

So $m = 6$ or $m = -\frac{6}{19}$. (Remark: if $m = 0$, the equation is $x^4 - 2x^2 = 0$, whose different roots $-\sqrt{2}, 0, \sqrt{2}$ also form an arithmetic sequence, although 0 is a double root.)

7. Find the value of the expression

$$\sin(\sin \alpha) \cdot \cos(\sin \beta) + \cos(\sin \alpha) \cdot \sin(\sin \beta)$$

if

$$180^\circ < \frac{\alpha + \beta}{2} < 270^\circ, \quad 90^\circ < \frac{\alpha - \beta}{2} < 180^\circ, \quad \tan \frac{\alpha + \beta}{2} = 1, \quad \text{and} \quad \tan \frac{\alpha - \beta}{2} = -2\sqrt{2}$$

Solution: By the addition formula, the expression is $\sin(\sin \alpha + \sin \beta)$. We use the sum-to-product identity

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

Since $\frac{\alpha + \beta}{2}$ is in the third quadrant and its tangent is 1, it is 225° , so $\sin \frac{\alpha + \beta}{2} = -\frac{\sqrt{2}}{2}$.

Since $\frac{\alpha - \beta}{2}$ is in the second quadrant, its cosine is negative. From $1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$,

$$\cos^2 \frac{\alpha - \beta}{2} = \frac{1}{1 + 8} = \frac{1}{9} \quad \text{so} \quad \cos \frac{\alpha - \beta}{2} = -\frac{1}{3}$$

So

$$\sin \alpha + \sin \beta = 2 \cdot \left(-\frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{1}{3}\right) = \frac{\sqrt{2}}{3}$$

and the value of the expression is $\sin \frac{\sqrt{2}}{3} \approx 0.454$ (here $\frac{\sqrt{2}}{3}$ is in radians).

8. Find the point P in the plane of triangle ABC for which $PA^2 + PB^2 + PC^2$ is minimal.

Solution: We use coordinates: let $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$, and $P(x, y)$. The terms containing x are

$$\begin{aligned}(x - a_1)^2 + (x - b_1)^2 + (x - c_1)^2 &= 3x^2 - 2(a_1 + b_1 + c_1)x + a_1^2 + b_1^2 + c_1^2 \\ &= 3\left(x - \frac{a_1 + b_1 + c_1}{3}\right)^2 + (\text{a constant})\end{aligned}$$

and this is minimal when $x = \frac{a_1 + b_1 + c_1}{3}$. The same way, the terms containing y are minimal when $y = \frac{a_2 + b_2 + c_2}{3}$.

So $PA^2 + PB^2 + PC^2$ is minimal when

$$P = \left(\frac{a_1 + b_1 + c_1}{3}, \frac{a_2 + b_2 + c_2}{3} \right)$$

that is, when P is the centroid of the triangle.