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translated from: KoMaL, February 1982, pages 61-62

Pdf file: KoMaL, February 1982, pages 61-62

The problems in this set are similar to the ones that students applying to universities and colleges usually solve at their entrance examinations. We recommend them to everyone who is applying. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation over the set of rational numbers.

$$\frac{2}{1-2x} + \frac{3}{2x+1} + \frac{6}{4x^2-1} = 1$$

Solution: The denominators cannot be zero, so $x \neq \pm \frac{1}{2}$. Notice that $\frac{2}{1-2x} = -\frac{2}{2x-1}$ and $4x^2-1 = (2x-1)(2x+1)$. We multiply both sides by $4x^2-1$.

$$-2(2x+1) + 3(2x-1) + 6 = 4x^2-1$$

$$2x+1 = 4x^2-1$$

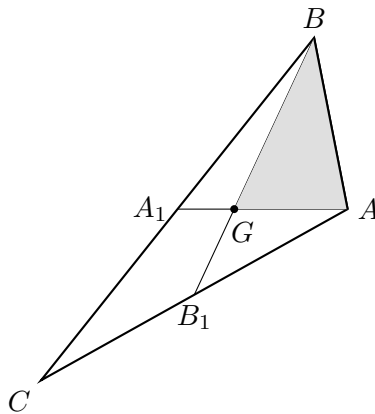
$$0 = 4x^2-2x-2 \quad \text{divide by 2}$$

$$0 = 2x^2-x-1 = (2x+1)(x-1)$$

So $x = 1$ or $x = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the domain. The only solution is $x = 1$. Check: $\frac{2}{-1} + \frac{3}{3} + \frac{6}{3} = -2+1+2 = 1$.

2. Two medians of a triangle are 3 and 5 units long, and the angle between them is 65° . Find the area of the triangle.

Solution: Let the medians be AA_1 and BB_1 , and let G be the centroid. The centroid divides each median in a 2 : 1 ratio, so $AG = \frac{2}{3} \cdot 3 = 2$ and $BG = \frac{2}{3} \cdot 5 = \frac{10}{3}$.



The angle between the medians at G is 65° or 115° , and these have the same sine. So the area of triangle ABG is

$$\frac{1}{2} \cdot 2 \cdot \frac{10}{3} \sin 65^\circ = \frac{10}{3} \sin 65^\circ$$

The three medians divide a triangle into six triangles of equal area, and triangle ABG consists of two of them. So the area of ABG is one third of the area of triangle ABC , and

$$A_{ABC} = 3 \cdot \frac{10}{3} \sin 65^\circ = 10 \sin 65^\circ \approx 9.06 \text{ unit}^2$$

3. The sum of three numbers is 71. If we increase the third number by 1, we get three consecutive terms of an arithmetic sequence. If instead we increase the first number by 3, we get three consecutive terms of a geometric sequence. Find the three numbers.

Solution: Let the numbers be a , b , and c . Then

$$a + b + c = 71, \quad 2b = a + (c + 1), \quad b^2 = (a + 3)c$$

From the first two equations, $2b = (71 - b) + 1$, so $b = 24$ and $a + c = 47$. Then

$$\begin{aligned} 576 &= (50 - c)c \\ c^2 - 50c + 576 &= 0 \\ c &= \frac{50 \pm \sqrt{2500 - 2304}}{2} = \frac{50 \pm 14}{2} \end{aligned}$$

So $c = 32$ or $c = 18$, and there are two solutions:

- 15, 24, 32: then 15, 24, 33 is arithmetic and 18, 24, 32 is geometric (with ratio $\frac{4}{3}$);
- 29, 24, 18: then 29, 24, 19 is arithmetic and 32, 24, 18 is geometric (with ratio $\frac{3}{4}$).

4. The center of a circle is on the line through the points $A(1, 1)$ and $B(7, 10)$. The circle is tangent to the line g with equation $x + 2y = 21$, at the point P whose x -coordinate is 5. Find the equation of the circle.

Solution: The point of tangency: with $x = 5$, $5 + 2y = 21$, so $y = 8$ and $P(5, 8)$. The radius drawn to the point of tangency is perpendicular to g , so the center is on the line through P perpendicular to g . A normal vector of g is $(1, 2)$, so the center is $(5 + t, 8 + 2t)$ for some t .

The line AB has slope $\frac{10 - 1}{7 - 1} = \frac{3}{2}$, and its equation is $3x - 2y = 1$. The center is on this line:

$$3(5 + t) - 2(8 + 2t) = 1 \quad \text{so} \quad -1 - t = 1 \quad \text{and} \quad t = -2$$

So the center is $(3, 4)$, and the radius is the distance between the center and P : $r = \sqrt{2^2 + 4^2} = \sqrt{20}$. The equation of the circle is

$$(x - 3)^2 + (y - 4)^2 = 20$$

5. Solve the given equation over the real numbers.

$$\frac{2 \sin x}{2 \sin x - 1} + \frac{2 \sin x - 1}{2 \sin x} = \frac{5}{2}$$

Solution: We need $\sin x \neq 0$ and $\sin x \neq \frac{1}{2}$. Let $u = \frac{2 \sin x}{2 \sin x - 1}$. Then the second term is $\frac{1}{u}$, and

$$\begin{aligned} u + \frac{1}{u} &= \frac{5}{2} \quad \text{multiply by } 2u \\ 2u^2 - 5u + 2 &= 0 \\ (2u - 1)(u - 2) &= 0 \end{aligned}$$

Let $w = 2 \sin x$. If $u = 2$, then $w = 2(w - 1)$, so $w = 2$ and $\sin x = 1$. If $u = \frac{1}{2}$, then $2w = w - 1$, so $w = -1$ and $\sin x = -\frac{1}{2}$. So the solutions are

$$x = \frac{\pi}{2} + 2k\pi, \quad x = \frac{7\pi}{6} + 2k\pi, \quad x = \frac{11\pi}{6} + 2k\pi$$

where k is any integer.

6. The base of a right prism is a triangle with sides 4 cm, 5 cm, and 7 cm. The height of the prism is equal to the longest altitude of the base triangle. Find the volume of the prism.

Solution: We find the area of the base with Heron's formula. Half of the perimeter is $s = 8$, so

$$T = \sqrt{8 \cdot (8 - 4)(8 - 5)(8 - 7)} = \sqrt{96} = 4\sqrt{6} \text{ cm}^2$$

The longest altitude belongs to the shortest side:

$$h = \frac{2T}{4} = 2\sqrt{6} \text{ cm}$$

So the volume of the prism is

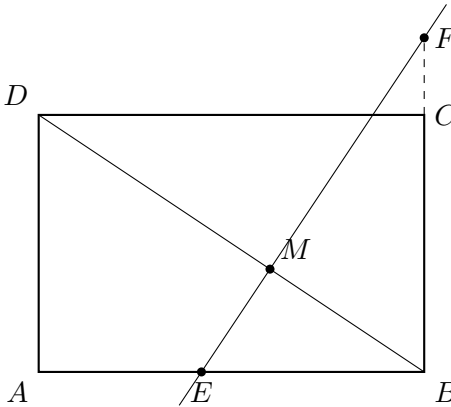
$$V = T \cdot h = 4\sqrt{6} \cdot 2\sqrt{6} = 48 \text{ cm}^3$$

7. Let M be a point inside the diagonal BD of rectangle $ABCD$. The line through M perpendicular to BD intersects line AB at E and line BC at F . Prove that

a) $AB^2 \cdot ME = BC^2 \cdot MF$;

b) the value of $\frac{AB \cdot ME + BC \cdot MF}{MB}$ does not depend on where M is on the segment BD .

Solution: Let $\varphi = \angle ABD$. Then $\angle DBC = 90^\circ - \varphi$, and in the right triangle ABD , $\tan \varphi = \frac{AD}{AB} = \frac{BC}{AB}$.



Triangle BME has a right angle at M and angle φ at B , and triangle BMF has a right angle at M and angle $90^\circ - \varphi$ at B . So

$$ME = MB \tan \varphi = MB \cdot \frac{BC}{AB} \quad \text{and} \quad MF = MB \tan (90^\circ - \varphi) = MB \cdot \frac{AB}{BC}$$

a) $AB^2 \cdot ME = AB^2 \cdot MB \cdot \frac{BC}{AB} = AB \cdot BC \cdot MB$, and $BC^2 \cdot MF = BC^2 \cdot MB \cdot \frac{AB}{BC} = AB \cdot BC \cdot MB$. So the two are equal.

b) $AB \cdot ME + BC \cdot MF = MB \cdot BC + MB \cdot AB = MB (AB + BC)$, and so

$$\frac{AB \cdot ME + BC \cdot MF}{MB} = AB + BC$$

which is half of the perimeter of the rectangle, and does not depend on M .

8. Find all real numbers x , y , and z that satisfy the given system of equations.

$$\begin{aligned}x + y + z &= 2 \\ 2xy - z^2 &= 4\end{aligned}$$

Solution: From the first equation, $z = 2 - x - y$. We substitute this into the second one.

$$\begin{aligned}2xy - (2 - x - y)^2 &= 4 \\ 2xy - (4 + x^2 + y^2 - 4x - 4y + 2xy) &= 4 \\ -x^2 - y^2 + 4x + 4y - 8 &= 0 \quad \text{multiply by } -1 \\ (x^2 - 4x + 4) + (y^2 - 4y + 4) &= 0 \\ (x - 2)^2 + (y - 2)^2 &= 0\end{aligned}$$

A sum of squares is zero only if each square is zero, so $x = 2$ and $y = 2$, and then $z = 2 - 2 - 2 = -2$. The only solution is $x = 2$, $y = 2$, $z = -2$. (Check: $2 \cdot 2 \cdot 2 - (-2)^2 = 4$.)