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The problems in this set are similar to the ones that students applying to universities and colleges usually solve at their entrance examinations. We recommend them to everyone who is applying. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given system of equations.

$$\begin{aligned}x^3 - x^2y - 4xy^2 + 4y^3 &= 0 \\ 2x^2 + y^2 &= 12\end{aligned}$$

Solution: We factor the left-hand side of the first equation by grouping.

$$x^3 - x^2y - 4xy^2 + 4y^3 = x^2(x - y) - 4y^2(x - y) = (x - y)(x - 2y)(x + 2y)$$

So $x = y$, or $x = 2y$, or $x = -2y$. We substitute each into the second equation.

If $x = y$: $3y^2 = 12$, so $y = \pm 2$, and we get $(2, 2)$ and $(-2, -2)$.

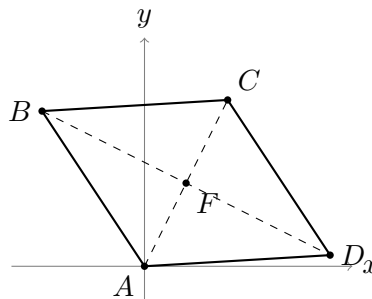
If $x = 2y$: $9y^2 = 12$, so $y = \pm \frac{2\sqrt{3}}{3}$, and we get $\left(\frac{4\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$ and $\left(-\frac{4\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$.

If $x = -2y$: again $9y^2 = 12$, and we get $\left(-\frac{4\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$ and $\left(\frac{4\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$.

So the system has six solutions.

2. Two opposite vertices of rhombus $ABCD$ are $A(0, 0)$ and $C(2, 4)$. The angles at vertices A and C measure 120° . Find the coordinates of vertices B and D .

Solution: The diagonal AC bisects the 120° angle at A , so $\angle BAC = 60^\circ$. Since $AB = BC$, triangle ABC is equilateral, and so $AB = AC = \sqrt{2^2 + 4^2} = 2\sqrt{5}$.



The diagonals of a rhombus bisect each other and are perpendicular. Their intersection is the midpoint of AC : $F(1, 2)$. In the right triangle AFB , $AF = \sqrt{5}$ and $FB = AF \tan 60^\circ = \sqrt{15}$. A unit vector perpendicular to $\overrightarrow{AC} = (2, 4)$ is $\frac{1}{\sqrt{5}}(-2, 1)$, so

$$B, D = (1, 2) \pm \sqrt{15} \cdot \frac{1}{\sqrt{5}}(-2, 1) = (1, 2) \pm \sqrt{3}(-2, 1)$$

So the two vertices are $(1 - 2\sqrt{3}, 2 + \sqrt{3})$ and $(1 + 2\sqrt{3}, 2 - \sqrt{3})$. Check: $(1 - 2\sqrt{3})^2 + (2 + \sqrt{3})^2 = 13 - 4\sqrt{3} + 7 + 4\sqrt{3} = 20$, so $AB = 2\sqrt{5}$.

3. Solve each of the following equations.

a) $(4 - x^2) \sqrt{1 - x} = 0$

b) $(0.4^x - 2.5^{x+1} - 1.5) \log_3 (x + 3) = 0$

c) $\sin x = 5 \cos \frac{x}{2}$

Solution: a) The domain is $x \leq 1$. A product is zero if one of its factors is zero. $4 - x^2 = 0$ gives $x = \pm 2$, but 2 is not in the domain; $\sqrt{1 - x} = 0$ gives $x = 1$. So the solutions are $x = -2$ and $x = 1$.

b) The domain is $x > -3$. The factor $\log_3 (x + 3)$ is zero if $x + 3 = 1$, that is, $x = -2$. For the other factor, let $u = 0.4^x$. Then $2.5^x = \frac{1}{u}$, since $2.5 = \frac{1}{0.4}$.

$$\begin{aligned} u - \frac{2.5}{u} - 1.5 &= 0 && \text{multiply by } 2u \\ 2u^2 - 3u - 5 &= 0 \\ (2u - 5)(u + 1) &= 0 \end{aligned}$$

Since $u > 0$, $u = 2.5$, that is, $0.4^x = 2.5 = 0.4^{-1}$, so $x = -1$. The solutions are $x = -2$ and $x = -1$.

c) Since $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$,

$$2 \sin \frac{x}{2} \cos \frac{x}{2} - 5 \cos \frac{x}{2} = 0 \quad \text{or} \quad \cos \frac{x}{2} \left(2 \sin \frac{x}{2} - 5 \right) = 0$$

The second factor is never zero, since $\sin \frac{x}{2} \leq 1 < \frac{5}{2}$. So $\cos \frac{x}{2} = 0$, $\frac{x}{2} = \frac{\pi}{2} + k\pi$, and

$$x = \pi + 2k\pi \quad \text{where } k \text{ is any integer.}$$

4. The perimeter of parallelogram $ABCD$ is 26 units, and $AB < AD$. The radius of the circle inscribed in triangle BCD is $\sqrt{3}$ units, and the angle at vertex B is 120° . Find the sides of the parallelogram.

Solution: Let $AB = CD = a$ and $AD = BC = b$, with $a < b$. Then $a + b = 13$. The angle at B is 120° , so the angle at C is 60° . In triangle BCD , the sides around the 60° angle are a and b , so by the Law of Cosines,

$$BD^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab = (a + b)^2 - 3ab = 169 - 3ab$$

The area of a triangle is rs , where r is the radius of the inscribed circle and s is half of the perimeter. So

$$\frac{1}{2}ab \sin 60^\circ = \sqrt{3} \cdot \frac{a + b + BD}{2} \quad \text{so} \quad \frac{\sqrt{3}}{4}ab = \frac{\sqrt{3}}{2}(13 + BD) \quad \text{and} \quad BD = \frac{ab}{2} - 13$$

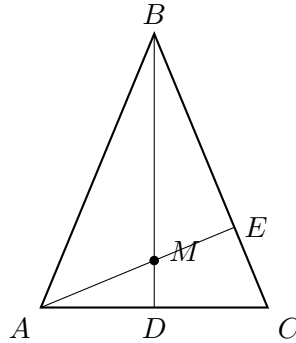
Let $p = ab$. Then

$$\begin{aligned} \left(\frac{p}{2} - 13 \right)^2 &= 169 - 3p \\ \frac{p^2}{4} - 13p + 169 &= 169 - 3p \\ \frac{p^2}{4} - 10p &= 0 \\ p(p - 40) &= 0 \end{aligned}$$

Since $p > 0$, $ab = 40$ and $a + b = 13$. So a and b are the roots of $t^2 - 13t + 40 = 0$, that is, 5 and 8. Since $a < b$, $AB = 5$ units and $AD = 8$ units. (Check: $BD^2 = 25 + 64 - 40 = 49$, so $BD = 7 = 20 - 13$.)

5. Triangle ABC is an acute isosceles triangle with $AB = BC$. Its orthocenter is M , and $MB = AC = 1$ unit. Find the angles of the triangle and the length of the legs. (How many solutions does the problem have if we do not assume that the triangle is acute?)

Solution: Let D be the midpoint of AC . Since the triangle is isosceles, BD is an altitude, and $AD = \frac{1}{2}$. Let α denote the angle at A (which is also the angle at C).



The orthocenter M is on BD and on the altitude AE drawn from A . In the right triangle AEC , the angle at A is $90^\circ - \alpha$. So in the right triangles ADM and ADB ,

$$MD = AD \tan(90^\circ - \alpha) = \frac{1}{2} \cot \alpha \quad \text{and} \quad BD = AD \tan \alpha = \frac{1}{2} \tan \alpha$$

If the triangle is acute, then M is inside the triangle, between B and D . Let $t = \tan \alpha$.

$$\begin{aligned} MB = BD - MD &= 1 \\ \frac{1}{2} \left(t - \frac{1}{t} \right) &= 1 && \text{multiply by } 2t \\ t^2 - 2t - 1 &= 0 \\ t &= 1 + \sqrt{2} \quad \text{since } t > 0 \end{aligned}$$

Since $\tan 67.5^\circ = 1 + \sqrt{2}$, the angles at A and C are 67.5° , and the angle at B is 45° . The legs are

$$AB = BC = \frac{AD}{\cos \alpha} = \frac{1}{2 \cos 67.5^\circ} = \frac{1}{\sqrt{2} - \sqrt{2}} \approx 1.307 \text{ units}$$

If we do not assume that the triangle is acute, then the angle at B could be obtuse (the base angles of an isosceles triangle are always acute). Then M is outside the triangle, on the ray DB beyond B , and $MB = MD - BD$. This gives $t^2 + 2t - 1 = 0$, $t = \sqrt{2} - 1 = \tan 22.5^\circ$. So the base angles are 22.5° , the angle at B is 135° , and the legs are $\frac{1}{2 \cos 22.5^\circ} = \frac{1}{\sqrt{2} + \sqrt{2}} \approx 0.541$ units. (If the angle at B were a right angle, then $M = B$ and $MB = 0$.) So without that assumption, the problem has two solutions.

6. Prove that in every triangle,

$$\sin^2 \frac{\alpha - \beta}{2} + \sin \alpha \sin \beta + \sin^2 \frac{\gamma}{2} = 1$$

where α , β , and γ are the angles of the triangle.

Solution: Since $\alpha + \beta + \gamma = 180^\circ$, we have $\frac{\gamma}{2} = 90^\circ - \frac{\alpha + \beta}{2}$, and so $\sin^2 \frac{\gamma}{2} = \cos^2 \frac{\alpha + \beta}{2}$. We use the identities

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}, \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

Then the left-hand side is

$$\frac{1 - \cos(\alpha - \beta)}{2} + \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} + \frac{1 + \cos(\alpha + \beta)}{2} = \frac{2}{2} = 1$$

7. Solve the given inequality.

$$\log_{x-3}(x-1) < 2$$

Solution: The base of the logarithm must be positive and not 1, and $x - 1 > 0$. So $x > 3$ and $x \neq 4$. We write 2 as a logarithm: $2 = \log_{x-3}(x-3)^2$. Notice that

$$(x-3)^2 - (x-1) = x^2 - 7x + 10 = (x-2)(x-5)$$

Case 1. If $x > 4$, the base is greater than 1, and the logarithm function is increasing. So we need $x - 1 < (x - 3)^2$, that is, $(x - 2)(x - 5) > 0$. With $x > 4$, this means $x > 5$.

Case 2. If $3 < x < 4$, the base is between 0 and 1, and the logarithm function is decreasing, so the inequality reverses: $x - 1 > (x - 3)^2$, that is, $(x - 2)(x - 5) < 0$, or $2 < x < 5$. This is true for every x with $3 < x < 4$.

So the solution is $3 < x < 4$ or $x > 5$.

8. Find all pairs of numbers (x, y) that satisfy each of the following equations.

a) $\sin^2(x + y) - \cos^2(x - y) = 1$

b) $\frac{1}{\sqrt{2}}(\sin x + \cos x) = 2y^2 - 4y + 3$

Solution: a) Since $\sin^2(x + y) \leq 1$ and $\cos^2(x - y) \geq 0$, the left-hand side is at most 1, and it is equal to 1 only if $\sin^2(x + y) = 1$ and $\cos^2(x - y) = 0$. So

$$x + y = \frac{\pi}{2} + k\pi \quad \text{and} \quad x - y = \frac{\pi}{2} + n\pi$$

for some integers k and n . Adding and subtracting,

$$x = \frac{\pi}{2} + (k + n)\frac{\pi}{2} \quad \text{and} \quad y = (k - n)\frac{\pi}{2}$$

Since $k + n$ and $k - n$ are both even or both odd, the solutions are

$$x = \frac{\pi}{2} + j\pi, \quad y = l\pi \quad \text{or} \quad x = j\pi, \quad y = \frac{\pi}{2} + l\pi$$

where j and l are any integers.

b) The left-hand side is

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} = \sin\left(x + \frac{\pi}{4}\right) \leq 1$$

and the right-hand side is $2y^2 - 4y + 3 = 2(y - 1)^2 + 1 \geq 1$. So the two sides are equal only if both are 1: $y = 1$ and $\sin\left(x + \frac{\pi}{4}\right) = 1$. The solutions are

$$x = \frac{\pi}{4} + 2k\pi, \quad y = 1 \quad \text{where } k \text{ is any integer.}$$