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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Is it true that $(4\sqrt{3} - 3)(\sqrt{3} + \sqrt{81} + \sqrt{27})$ is greater than $4^{1+\log_2 3}$?

Solution: Since $\sqrt{81} = 9$ and $\sqrt{27} = 3\sqrt{3}$, the second factor is $9 + 4\sqrt{3}$, and

$$(4\sqrt{3} - 3)(4\sqrt{3} + 9) = 48 + 36\sqrt{3} - 12\sqrt{3} - 27 = 21 + 24\sqrt{3} \approx 62.57$$

On the other hand,

$$4^{1+\log_2 3} = 4 \cdot 4^{\log_2 3} = 4 \cdot (2^{\log_2 3})^2 = 4 \cdot 9 = 36$$

So yes, it is true: $21 + 24\sqrt{3} > 36$.

2. A triangle is inscribed in a circle of radius 4 cm. One of its sides is 6 cm, and one of its angles is 20° . How long is its longest side?

Solution: By the Law of Sines, each side is $2R = 8$ times the sine of the opposite angle. The side opposite the 20° angle is $8 \sin 20^\circ \approx 2.74 \neq 6$, so the 6 cm side is opposite another angle θ , with

$$\sin \theta = \frac{6}{8} = \frac{3}{4} \quad \text{so} \quad \theta \approx 48.59^\circ \quad \text{or} \quad \theta \approx 131.41^\circ$$

Both are possible, so there are two triangles.

If $\theta \approx 48.59^\circ$, the third angle is about 111.41° , and the longest side is opposite it:

$$8 \sin (20^\circ + \theta) = 8 \left(\sin 20^\circ \cdot \frac{\sqrt{7}}{4} + \cos 20^\circ \cdot \frac{3}{4} \right) = 2\sqrt{7} \sin 20^\circ + 6 \cos 20^\circ \approx 7.45 \text{ cm}$$

(Here we used $\sin (180^\circ - x) = \sin x$ and $\cos \theta = \frac{\sqrt{7}}{4}$.)

If $\theta \approx 131.41^\circ$, the third angle is about 28.59° , and the longest side is the 6 cm side itself.

So the longest side is about 7.45 cm in the first triangle and 6 cm in the second one.

3. Find the smallest positive root of the equation $\sin(x - 1) = \sin(\pi - 1)$.

Solution: Since $\sin(\pi - 1) = \sin 1$, the equation is $\sin(x - 1) = \sin 1$. So

$$x - 1 = 1 + 2k\pi \quad \text{or} \quad x - 1 = \pi - 1 + 2k\pi$$

that is, $x = 2 + 2k\pi$ or $x = \pi + 2k\pi$, where k is any integer. The smallest positive root is $x = 2$ (since $2 < \pi$).

4. Two space diagonals of a right prism with a regular hexagon base are 24 cm and 25 cm long. Find the volume of the prism.

Solution: Let the side of the base be a and the height h . A space diagonal connects a vertex of the bottom with a vertex of the top that is not directly above it. Its projection onto the base is a diagonal of the hexagon, which is either $a\sqrt{3}$ (short diagonal) or $2a$ (long diagonal). So

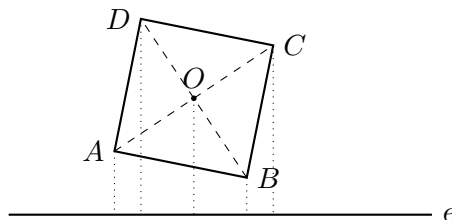
$$3a^2 + h^2 = 24^2 = 576 \quad \text{and} \quad 4a^2 + h^2 = 25^2 = 625$$

Subtracting, $a^2 = 49$, so $a = 7$ and $h^2 = 576 - 147 = 429$, $h = \sqrt{429}$. The area of the base is $6 \cdot \frac{\sqrt{3}}{4}a^2 = \frac{147\sqrt{3}}{2}$, and the volume is

$$V = \frac{147\sqrt{3}}{2} \cdot \sqrt{429} = \frac{441\sqrt{143}}{2} \approx 2637 \text{ cm}^3$$

5. The square $ABCD$ has no common point with a line e in its plane. The distances of the vertices of the square from e are a, b, c , and d , respectively. Prove that $a + c = b + d$.

Solution: Since the square and the line have no common point, the whole square is on one side of e . Let O be the center of the square, and let o be its distance from e .



The diagonals of the square bisect each other, so O is the midpoint of AC and of BD . The perpendiculars from A and C to e are parallel, so they form a trapezoid (or a rectangle) with e and AC , and the perpendicular from O is its midline. The midline of a trapezoid is the average of the parallel sides, so $o = \frac{a+c}{2}$. The same way, $o = \frac{b+d}{2}$. So $a + c = 2o = b + d$.

6. For which values of a does the given equation have a root?

$$(8 - 2a - a^2) \cot x = \tan x$$

Solution: We need $\sin x \neq 0$ and $\cos x \neq 0$, so $\tan x \neq 0$. Multiplying both sides by $\tan x$ (and using $\cot x \cdot \tan x = 1$),

$$\tan^2 x = 8 - 2a - a^2$$

This has a solution with $\tan x \neq 0$ if and only if the right-hand side is positive:

$$a^2 + 2a - 8 < 0 \quad (a + 4)(a - 2) < 0$$

So the equation has a root if and only if $-4 < a < 2$.

7. Solve the given equation.

$$\sqrt{x-3} + \sqrt{7x-1} = 7-x$$

Solution: For the square roots we need $x \geq 3$. Then $\sqrt{x-3} \geq 0$ and $\sqrt{7x-1} \geq \sqrt{20} > 4$, so the left-hand side is greater than 4. But the right-hand side is $7-x \leq 4$. So the two sides are never equal, and the equation has no solution.

8. Prove that if n is an integer, then $\frac{7n-1}{4}$ and $\frac{5n+3}{12}$ cannot both be integers.

Solution: Suppose that both are integers. If $\frac{5n+3}{12}$ is an integer, then $5n+3$ is divisible by 12, and so by 4. Since $5n+3 = 4n + (n+3)$, this means that $n+3$ is divisible by 4, so n gives a remainder of 1 when divided by 4.

If $\frac{7n-1}{4}$ is an integer, then $7n-1 = 8n - (n+1)$ is divisible by 4, so $n+1$ is divisible by 4, and n gives a remainder of 3 when divided by 4.

The remainder cannot be both 1 and 3, so the two numbers cannot both be integers.