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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The base of an isosceles triangle is 24 units long, and its legs are 20 units long. How far is the centroid of the triangle from the center of its circumscribed circle?

Solution: We place the triangle in a coordinate system: $A(-12, 0)$, $B(12, 0)$, and $C(0, h)$, where $h = \sqrt{20^2 - 12^2} =$

16. The centroid is $G\left(0, \frac{16}{3}\right)$.

The center O of the circumscribed circle is on the axis of symmetry, $O(0, k)$, and $OA = OC$:

$$\begin{aligned} 12^2 + k^2 &= (16 - k)^2 \\ 144 + k^2 &= 256 - 32k + k^2 \\ 32k &= 112 \\ k &= \frac{7}{2} \end{aligned}$$

So the distance is $OG = \frac{16}{3} - \frac{7}{2} = \frac{11}{6}$ units.

2. Solve each of the following equations.

a) $\sqrt{\frac{x}{1+x}} + \sqrt{\frac{1+x}{x}} = \frac{5}{2}$

b) $\cot x + \frac{1}{\sin x} = \tan x$

c) $2 \log_4 (4 - x) = 4 - \log_2 (-x - 2)$

Solution: a) We need $\frac{x}{1+x} > 0$, that is, $x < -1$ or $x > 0$. Let $t = \sqrt{\frac{x}{1+x}} > 0$. Then the equation is $t + \frac{1}{t} = \frac{5}{2}$, that is, $2t^2 - 5t + 2 = 0$, so $t = 2$ or $t = \frac{1}{2}$.

If $t = 2$: $\frac{x}{1+x} = 4$, so $x = 4 + 4x$ and $x = -\frac{4}{3}$.

If $t = \frac{1}{2}$: $\frac{x}{1+x} = \frac{1}{4}$, so $4x = 1 + x$ and $x = \frac{1}{3}$.

Both are in the domain, so the solutions are $x = -\frac{4}{3}$ and $x = \frac{1}{3}$.

b) We need $\sin x \neq 0$ and $\cos x \neq 0$. We multiply both sides by $\sin x \cos x$.

$$\begin{aligned}\cos^2 x + \cos x &= \sin^2 x & \sin^2 x &= 1 - \cos^2 x \\ 2 \cos^2 x + \cos x - 1 &= 0 \\ (2 \cos x - 1)(\cos x + 1) &= 0\end{aligned}$$

If $\cos x = -1$, then $\sin x = 0$, which is not allowed. So $\cos x = \frac{1}{2}$, and the solutions are $x = \pm \frac{\pi}{3} + 2k\pi$, where k is any integer.

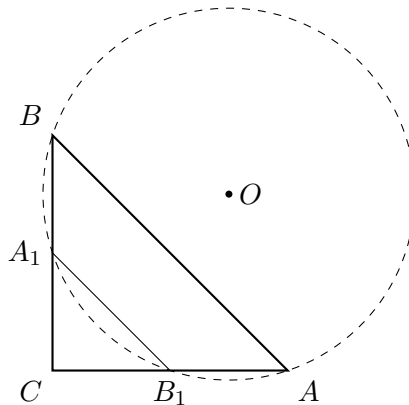
c) We need $4 - x > 0$ and $-x - 2 > 0$, that is, $x < -2$. Since $2 \log_4 (4 - x) = \log_2 (4 - x)$,

$$\begin{aligned}\log_2 (4 - x) + \log_2 (-x - 2) &= 4 \\ (4 - x)(-x - 2) &= 16 \\ x^2 - 2x - 24 &= 0 \\ (x - 6)(x + 4) &= 0\end{aligned}$$

Only $x = -4$ is in the domain. Check: $2 \log_4 8 = 3$ and $4 - \log_2 2 = 3$.

3. The hypotenuse of the right triangle ABC is $AB = 2$ units. Let B_1 be the midpoint of leg AC , and A_1 the midpoint of leg BC . Find the radius of the circle circumscribed around the quadrilateral ABA_1B_1 , if this quadrilateral is cyclic.

Solution: The segment A_1B_1 is a midline of the triangle, so it is parallel to AB , and ABA_1B_1 is a trapezoid. A trapezoid is cyclic only if it is isosceles, so $AB_1 = BA_1$, that is, $AC = BC$. So the triangle is an isosceles right triangle, and its legs are $\sqrt{2}$.



We use coordinates: $C(0, 0)$, $A(\sqrt{2}, 0)$, $B(0, \sqrt{2})$, $B_1\left(\frac{\sqrt{2}}{2}, 0\right)$, and $A_1\left(0, \frac{\sqrt{2}}{2}\right)$. By symmetry, the center is on the line $y = x$, say $O(t, t)$. The center is on the perpendicular bisector of AB_1 , which is the line $x = \frac{1}{2}\left(\sqrt{2} + \frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{4}$. So $t = \frac{3\sqrt{2}}{4}$, and

$$R = OA = \sqrt{\left(\sqrt{2} - \frac{3\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4}\right)^2} = \sqrt{\frac{2}{16} + \frac{18}{16}} = \frac{\sqrt{5}}{2} \approx 1.118 \text{ units}$$

4. Consider the equation

$$(4x^2 - 4(a-2)x + a)(4x^2 - 4ax + 3a - 2) = 0$$

Find all values of the real parameter a for which every root of this equation is real.

Solution: Every root is real if and only if both quadratic factors have real roots, that is, both discriminants are non-negative.

$$D_1 = 16(a-2)^2 - 16a = 16(a^2 - 5a + 4) = 16(a-1)(a-4) \geq 0$$

$$D_2 = 16a^2 - 16(3a-2) = 16(a^2 - 3a + 2) = 16(a-1)(a-2) \geq 0$$

The first condition holds if $a \leq 1$ or $a \geq 4$, and the second one if $a \leq 1$ or $a \geq 2$. Both hold if and only if $a \leq 1$ or $a \geq 4$.

5. Find the equation of the circle that is tangent to the line $2x + y = 19$ at the point with x -coordinate 7, and cuts chords of equal length from the two coordinate axes.

Solution: The point of tangency: with $x = 7$, $y = 19 - 14 = 5$, so $T(7, 5)$. The center is on the line through T perpendicular to the tangent line; a normal vector of $2x + y = 19$ is $(2, 1)$, so the center is $(7 + 2t, 5 + t)$ for some t .

A circle cuts chords of equal length from the two axes if and only if its center has the same distance from the two axes (a chord of length $2\sqrt{r^2 - d^2}$ belongs to distance d). So $|7 + 2t| = |5 + t|$.

If $7 + 2t = 5 + t$, then $t = -2$, the center is $(3, 3)$, and $r^2 = 4^2 + 2^2 = 20$.

If $7 + 2t = -(5 + t)$, then $t = -4$, the center is $(-1, 1)$, and $r^2 = 8^2 + 4^2 = 80$.

So there are two such circles:

$$(x-3)^2 + (y-3)^2 = 20 \quad \text{and} \quad (x+1)^2 + (y-1)^2 = 80$$

Both circles intersect both axes, since the distance of the center from each axis is less than the radius.

6. The perimeter of a right trapezoid is 4 units, and its acute angle is 45° . How should we choose the height of the trapezoid so that its area is as large as possible? What is the maximal area?

Solution: Let a and c be the parallel sides ($a > c$) and h the height. The leg perpendicular to the bases is h . Since the acute angle is 45° , the other leg is $h\sqrt{2}$ and $a - c = h$. From the perimeter,

$$a + c + h + \sqrt{2}h = 4 \quad \text{so} \quad a + c = 4 - (1 + \sqrt{2})h$$

and the area is

$$T(h) = \frac{a+c}{2} \cdot h = \frac{h(4 - (1 + \sqrt{2})h)}{2}$$

This is a quadratic function of h with roots 0 and $\frac{4}{1 + \sqrt{2}}$, so its maximum is halfway between them:

$$h = \frac{2}{1 + \sqrt{2}} = 2(\sqrt{2} - 1) \approx 0.83$$

Then $a + c = 2$, and the maximal area is $T = \frac{2}{2} \cdot h = 2\sqrt{2} - 2 \approx 0.83$ square units. (Here $c = \frac{2-h}{2} > 0$, so such a trapezoid exists.)

7. The sides of a triangle are a , b , and c . Prove that if

$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

then the angle opposite side b is 60° . State the converse of this statement and prove it as well.

Solution: Let $s = a + c$. Since all denominators are positive, the equation is equivalent to the one we get by multiplying by $(a+b)(b+c)(a+b+c)$:

$$\begin{aligned}(a+2b+c)(a+b+c) &= 3(a+b)(b+c) \\(s+2b)(s+b) &= 3(bs+ac+b^2) \\s^2+3bs+2b^2 &= 3bs+3ac+3b^2 \\s^2-3ac &= b^2 \\a^2+c^2-ac &= b^2\end{aligned}$$

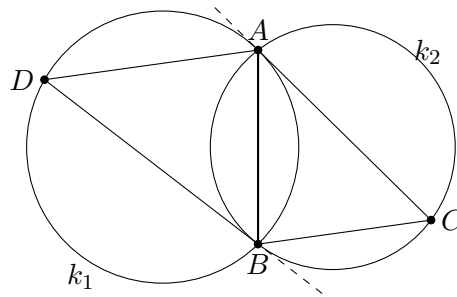
By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac \cos \beta$, where β is the angle opposite side b . So the equation holds if and only if $2 \cos \beta = 1$, that is, $\beta = 60^\circ$.

The converse: if the angle opposite side b is 60° , then $\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$. This is also true, since every step above can be reversed.

8. The circles k_1 and k_2 intersect at points A and B . The tangent line to k_1 at A intersects k_2 at C , and the tangent line to k_2 at B intersects k_1 at D . Prove that

$$AB^2 = AD \cdot BC \quad \text{and} \quad BD^2 \cdot BC = AC^2 \cdot AD$$

Solution:



The angle between a tangent and a chord equals the inscribed angle on the other side of the chord. Line AC is tangent to k_1 at A , and D is on k_1 , so $\angle CAB = \angle ADB$. Line BD is tangent to k_2 at B , and C is on k_2 , so $\angle DBA = \angle BCA$.

So triangles DBA and ACB have two pairs of equal angles: the angle at D equals the angle at A , and the angle at B equals the angle at C . The triangles are similar, with $D \leftrightarrow A$, $B \leftrightarrow C$, $A \leftrightarrow B$, and so

$$\frac{DB}{AC} = \frac{BA}{CB} = \frac{AD}{BA}$$

From the second equality, $AB^2 = AD \cdot BC$. Multiplying the first ratio by itself, using the other two,

$$\left(\frac{DB}{AC}\right)^2 = \frac{BA}{CB} \cdot \frac{AD}{BA} = \frac{AD}{BC} \quad \text{so} \quad BD^2 \cdot BC = AC^2 \cdot AD$$