

author: Róbert Freud

translated from: KoMaL, February 1983, pages 62-63

Pdf file: KoMaL, February 1983, pages 62-63

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation.

$$\frac{\sqrt{x} + x}{\sqrt{x} - x} + \frac{\sqrt{x} - x}{\sqrt{x} + x} = \frac{4}{x} + \frac{4}{1 - x}$$

Solution: We need  $x > 0$  (because of  $\sqrt{x}$  and  $\frac{4}{x}$ ),  $\sqrt{x} \neq x$ , that is,  $x \neq 1$ . Let  $s = \sqrt{x}$ , and simplify the fractions on the left-hand side by  $s$ :

$$\frac{s + s^2}{s - s^2} + \frac{s - s^2}{s + s^2} = \frac{1 + s}{1 - s} + \frac{1 - s}{1 + s} = \frac{(1 + s)^2 + (1 - s)^2}{1 - s^2} = \frac{2 + 2s^2}{1 - s^2} = \frac{2(1 + x)}{1 - x}$$

The right-hand side is  $\frac{4(1 - x) + 4x}{x(1 - x)} = \frac{4}{x(1 - x)}$ . Multiplying both sides by  $x(1 - x) \neq 0$ ,

$$2x(1 + x) = 4$$

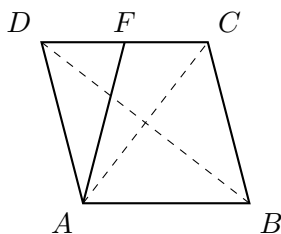
$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

But  $x = -2$  and  $x = 1$  are both outside the domain. So the equation has no solution.

2. We connect a vertex of a rhombus with the midpoint of one of the opposite sides. This splits the rhombus into a triangle and an isosceles trapezoid. What is the ratio of the diagonals of the rhombus?

Solution: Let the rhombus be  $ABCD$  with sides 1, and let  $F$  be the midpoint of  $CD$ . The segment  $AF$  splits the rhombus into triangle  $ADF$  and quadrilateral  $ABCF$ . Since  $AB \parallel CF$ ,  $ABCF$  is a trapezoid, and since  $AB = 1 \neq \frac{1}{2} = CF$ , it is not a parallelogram. So it is isosceles exactly when its legs are equal:  $AF = BC = 1$ .



In triangle  $ADF$ ,  $AD = 1$ ,  $DF = \frac{1}{2}$ , and  $AF = 1$ . By the Law of Cosines, the angle  $\delta$  at  $D$  satisfies

$$\cos \delta = \frac{1^2 + \left(\frac{1}{2}\right)^2 - 1^2}{2 \cdot 1 \cdot \frac{1}{2}} = \frac{1}{4}$$

The diagonal  $AC$  is opposite the angle at  $D$  in triangle  $ACD$ , and the diagonal  $BD$  is opposite the angle at  $A$  in triangle  $ABD$ , where  $\cos A = -\cos \delta = -\frac{1}{4}$ . So

$$AC^2 = 2 - 2 \cdot \frac{1}{4} = \frac{3}{2}, \quad BD^2 = 2 + 2 \cdot \frac{1}{4} = \frac{5}{2}, \quad \text{and} \quad \frac{AC}{BD} = \sqrt{\frac{3}{5}}$$

The ratio of the diagonals is  $\sqrt{3} : \sqrt{5} \approx 0.775$ .

3. Solve the given equation.

$$\log_{2x}(6x^2 - x) = 3$$

Solution: The base must be positive and not 1:  $x > 0$  and  $x \neq \frac{1}{2}$ . The argument must be positive:  $x(6x - 1) > 0$ , so (since  $x > 0$ )  $x > \frac{1}{6}$ . By the definition of the logarithm,

$$\begin{aligned} 6x^2 - x &= (2x)^3 = 8x^3 \\ 0 &= 8x^3 - 6x^2 + x \\ 0 &= x(8x^2 - 6x + 1) = x(4x - 1)(2x - 1) \end{aligned}$$

Of  $x = 0, \frac{1}{4}, \frac{1}{2}$ , only  $x = \frac{1}{4}$  is in the domain. Check:  $\log_{1/2}\left(\frac{6}{16} - \frac{1}{4}\right) = \log_{1/2} \frac{1}{8} = 3$ .

4. In a right triangle, the altitude drawn from the right angle is on the line  $y = 3x + 2$ , and the median drawn from the right angle is on the line  $y = 2x + 3$ . The point  $(6, 8)$  is on the line of the hypotenuse. Find the coordinates of the vertices.

Solution: The right angle  $C$  is the intersection of the altitude and the median:  $3x + 2 = 2x + 3$ , so  $C(1, 5)$ .

The hypotenuse is perpendicular to the altitude, so its slope is  $-\frac{1}{3}$ , and it passes through  $(6, 8)$ :  $y = -\frac{1}{3}x + 10$ . The median meets the hypotenuse at its midpoint  $M$ :

$$2x + 3 = -\frac{1}{3}x + 10 \quad \text{so} \quad x = 3 \quad \text{and} \quad M(3, 9)$$

In a right triangle, the midpoint of the hypotenuse is the center of the circumscribed circle, so  $MA = MB = MC = \sqrt{2^2 + 4^2} = \sqrt{20}$ . A unit vector along the hypotenuse is  $\frac{1}{\sqrt{10}}(3, -1)$ , so

$$A, B = (3, 9) \pm \sqrt{20} \cdot \frac{1}{\sqrt{10}}(3, -1) = (3, 9) \pm \sqrt{2}(3, -1)$$

So the vertices are  $C(1, 5)$ ,  $(3 + 3\sqrt{2}, 9 - \sqrt{2})$ , and  $(3 - 3\sqrt{2}, 9 + \sqrt{2})$ . Check: the dot product of the vectors from  $C$  to the other two vertices is

$$(2 + 3\sqrt{2})(2 - 3\sqrt{2}) + (4 - \sqrt{2})(4 + \sqrt{2}) = -14 + 14 = 0$$

so the angle at  $C$  is a right angle.

5. Solve the given equation.

$$\tan x + \sqrt{3} \tan^2 x = \sqrt{3} \cot x + \cot^2 x$$

Solution: We need  $\sin x \neq 0$  and  $\cos x \neq 0$ . Let  $t = \tan x$ , so  $\cot x = \frac{1}{t}$ . Multiplying by  $t^2$ ,

$$\begin{aligned} t + \sqrt{3}t^2 &= \frac{\sqrt{3}}{t} + \frac{1}{t^2} \quad \text{multiply by } t^2 \\ \sqrt{3}t^4 + t^3 - \sqrt{3}t - 1 &= 0 \\ t^3(\sqrt{3}t + 1) - (\sqrt{3}t + 1) &= 0 \\ (\sqrt{3}t + 1)(t^3 - 1) &= 0 \end{aligned}$$

So  $\tan x = -\frac{1}{\sqrt{3}}$  or  $\tan x = 1$ , and the solutions are

$$x = -\frac{\pi}{6} + k\pi \quad \text{or} \quad x = \frac{\pi}{4} + k\pi$$

where  $k$  is any integer.

6. For which real numbers are the following functions defined? Also find the range of each function.

a)  $\cos(\cos x)$

b)  $\sqrt{\log_2(\sqrt{-2x} - \sqrt{-x})}$

Solution: a) The function is defined for every real number. The values of  $\cos x$  fill the interval  $[-1, 1]$ . On this interval,  $\cos t$  is even, and it is decreasing on  $[0, 1]$  (since  $1 < \frac{\pi}{2}$ ). So the range is  $[\cos 1, 1] \approx [0.540, 1]$ .

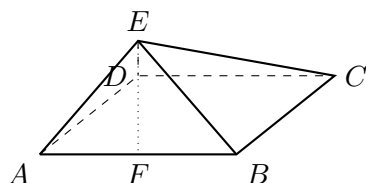
b) We need  $x \leq 0$  for the inner square roots. Since  $\sqrt{-2x} - \sqrt{-x} = (\sqrt{2} - 1)\sqrt{-x}$ , the logarithm is defined if  $x < 0$ . The outer square root needs  $\log_2((\sqrt{2} - 1)\sqrt{-x}) \geq 0$ :

$$(\sqrt{2} - 1)\sqrt{-x} \geq 1 \quad \text{so} \quad \sqrt{-x} \geq \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 \quad \text{and} \quad x \leq -(\sqrt{2} + 1)^2 = -3 - 2\sqrt{2}$$

So the domain is  $x \leq -3 - 2\sqrt{2}$ . At  $x = -3 - 2\sqrt{2}$  the value is 0, and as  $x$  decreases, the value increases without bound. So the range is  $[0, \infty)$ .

7. The base of a pyramid is a square, and its apex is directly above the midpoint of one of the base edges. One of the lateral edges is twice as long as another one. Find the angles of the lateral faces.

Solution: Let the base be  $ABCD$  with sides 2, and let the apex  $E$  be above the midpoint  $F$  of  $AB$ , at height  $m$ .



By symmetry  $EA = EB$  and  $EC = ED$ , and

$$EA^2 = 1 + m^2, \quad EC^2 = FC^2 + m^2 = 5 + m^2$$

Since  $EC > EA$ , we need  $EC = 2EA$ :  $5 + m^2 = 4 + 4m^2$ , so  $m^2 = \frac{1}{3}$ ,  $EA = \frac{2}{\sqrt{3}}$ , and  $EC = \frac{4}{\sqrt{3}}$ .

Face  $EAB$ : it is isosceles with legs  $\frac{2}{\sqrt{3}}$  and base 2. Its height is  $m = \frac{1}{\sqrt{3}}$ , so  $\tan \angle EAB = \frac{1}{\sqrt{3}}$ . Its angles are  $30^\circ$ ,  $30^\circ$ , and  $120^\circ$ .

Faces  $EBC$  and  $EAD$  (they are congruent):  $EB^2 + BC^2 = \frac{4}{3} + 4 = \frac{16}{3} = EC^2$ , so there is a right angle at  $B$ . (Indeed,  $BC$  is perpendicular to the plane  $EAB$ .) Then  $\tan \angle ECB = \frac{EB}{BC} = \frac{1}{\sqrt{3}}$ , and the angles are  $90^\circ$ ,  $30^\circ$ , and  $60^\circ$ .

Face  $ECD$ : it is isosceles with legs  $\frac{4}{\sqrt{3}}$  and base 2. The base angles satisfy  $\cos \angle ECD = \frac{1}{4/\sqrt{3}} = \frac{\sqrt{3}}{4}$ , so they are about  $64.3^\circ$ , and the angle at  $E$  is about  $51.3^\circ$  (its cosine is  $\frac{5}{8}$ ).

Remark: the angles between the lateral faces and the base are  $90^\circ$  (face  $EAB$ ),  $30^\circ$  (faces  $EBC$  and  $EAD$ ), and  $\arctan \frac{1}{2\sqrt{3}} \approx 16.1^\circ$  (face  $ECD$ ).

8. The four zeros of the parabolas  $y = x^2 + b_1x + c_1$  and  $y = x^2 + b_2x + c_2$  form an arithmetic sequence. Prove that the two discriminants are either equal or one of them is nine times the other.

Solution: The discriminant of  $x^2 + bx + c$  is  $b^2 - 4c$ . If the roots are  $r$  and  $s$ , then  $b = -(r + s)$  and  $c = rs$ , so

$$b^2 - 4c = (r + s)^2 - 4rs = (r - s)^2$$

that is, the discriminant is the square of the difference of the roots.

Let the four zeros be  $t, t + d, t + 2d, t + 3d$ . Each parabola has two of them, and there are three ways to split them into two pairs:

- $\{t, t + d\}$  and  $\{t + 2d, t + 3d\}$ : the discriminants are  $d^2$  and  $d^2$ ;
- $\{t, t + 2d\}$  and  $\{t + d, t + 3d\}$ : the discriminants are  $4d^2$  and  $4d^2$ ;
- $\{t, t + 3d\}$  and  $\{t + d, t + 2d\}$ : the discriminants are  $9d^2$  and  $d^2$ .

So the two discriminants are either equal, or one of them is nine times the other.