

author: Viktor Scharnitzky

translated from: KoMaL, March 1983, pages 105-106

Pdf file: KoMaL, March 1983, pages 105-106

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve each of the following equations.

a) $9\sqrt[3]{x} - 2 \cdot 3\sqrt[3]{x} = 3$

b) $\frac{100 - x^2}{10 - x} = |10 + x|$

c) $\frac{5}{3}\sqrt{15x} - \frac{3}{5}\sqrt{15x} - 11 = \frac{1}{3}\sqrt{15x}$

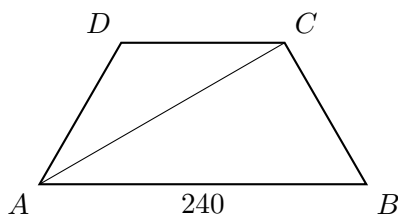
Solution: a) Let $u = 3\sqrt[3]{x} > 0$. Then $9\sqrt[3]{x} = u^2$, and the equation is $u^2 - 2u - 3 = 0$, that is, $(u - 3)(u + 1) = 0$. Since $u > 0$, $u = 3$, so $\sqrt[3]{x} = 1$ and $x = 1$.

b) We need $x \neq 10$. Then $\frac{100 - x^2}{10 - x} = \frac{(10 - x)(10 + x)}{10 - x} = 10 + x$, so the equation is $10 + x = |10 + x|$. This is true exactly when $10 + x \geq 0$. So the solutions are all $x \geq -10$ except $x = 10$.

c) Let $w = \sqrt{15x}$. Then $\left(\frac{5}{3} - \frac{3}{5} - \frac{1}{3}\right)w = 11$, that is, $\frac{25 - 9 - 5}{15}w = \frac{11}{15}w = 11$. So $w = 15$, $15x = 225$, and $x = 15$.

2. The longer one of the parallel sides of trapezoid $ABCD$ is $AB = 240$. The diagonal AC makes an angle of 30° with AB , it is perpendicular to side BC , and it bisects the angle DAB . Find the area of the trapezoid.

Solution: In triangle ABC , the angle at C is 90° and the angle at A is 30° , so $BC = \frac{AB}{2} = 120$, $AC = 120\sqrt{3}$, and the angle at B is 60° .



Since AC bisects angle DAB , the angle DAC is also 30° , and the angle at A is 60° . Since $AB \parallel CD$, $\angle DCA = \angle CAB = 30^\circ$. So triangle ACD has two 30° angles and $AD = DC$. The angles at A and B are both 60° , so the trapezoid is isosceles: $AD = BC = 120$, and so $CD = 120$ too. The height is $BC \sin 60^\circ = 60\sqrt{3}$, and the area is

$$T = \frac{240 + 120}{2} \cdot 60\sqrt{3} = 10\,800\sqrt{3} \approx 18\,706 \text{ unit}^2$$

3. The endpoints of one of the diagonals of rectangle $ABCD$ are $A(-10, -6)$ and $C(9, 16)$, and the slope of the line AB is $\frac{5}{12}$. Find the coordinates of vertices B and D , and the area of the rectangle.

Solution: Vertex B is on the line through A with slope $\frac{5}{12}$, and on the line through C perpendicular to it, with slope $-\frac{12}{5}$:

$$y + 6 = \frac{5}{12}(x + 10) \quad \text{and} \quad y - 16 = -\frac{12}{5}(x - 9)$$

Multiplying the difference of the two equations by 60: $25(x + 10) - 360 = -144(x - 9) + 960$, so $169x = 2366$, $x = 14$, and $y = 4$. So $B(14, 4)$. The diagonals bisect each other, so $D = A + C - B = (-15, 6)$.

The sides are $AB = \sqrt{24^2 + 10^2} = 26$ and $BC = \sqrt{5^2 + 12^2} = 13$, so the area is $26 \cdot 13 = 338$ square units.

4. The sum of the first three terms of an arithmetic sequence is 15. If we square these terms, we get three consecutive terms of a geometric sequence. Find the common difference of the arithmetic sequence and the common ratio of the geometric sequence.

Solution: Let the terms be $5 - d$, 5 , $5 + d$ (their sum is 15). Their squares form a geometric sequence if

$$\begin{aligned} 25^2 &= (5 - d)^2(5 + d)^2 \\ 625 &= (25 - d^2)^2 \\ 25 - d^2 &= \pm 25 \end{aligned}$$

So $d = 0$ or $d^2 = 50$, that is, $d = \pm 5\sqrt{2}$.

If $d = 0$, all terms are 5, and the common ratio is $q = 1$.

If $d = 5\sqrt{2}$, the squares are $25(1 - \sqrt{2})^2 = 25(3 - 2\sqrt{2})$, 25, and $25(3 + 2\sqrt{2})$, so $q = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}$.

If $d = -5\sqrt{2}$, the order is reversed, and $q = 3 - 2\sqrt{2}$.

5. In triangle ABC , $AC = BC$. We choose points D and E on side AC so that $AD = DE = EC$. Find the area of the triangle if $BD = 8.5$ and $BE = 10$.

Solution: Let $AD = DE = EC = t$, so $AC = BC = 3t$, and let $AB = c$. By Stewart's theorem, for a point P on side AC with $AP = p$,

$$BP^2 = \frac{AB^2(AC - p) + BC^2 \cdot p}{AC} - p(AC - p)$$

With $p = t$ and $p = 2t$,

$$\begin{aligned} BD^2 &= \frac{2c^2t + 9t^2 \cdot t}{3t} - 2t^2 = \frac{2c^2}{3} + t^2 = 72.25 \\ BE^2 &= \frac{c^2t + 9t^2 \cdot 2t}{3t} - 2t^2 = \frac{c^2}{3} + 4t^2 = 100 \end{aligned}$$

Subtracting the first equation from twice the second, $7t^2 = 127.75$, so $t^2 = 18.25$; then $\frac{c^2}{3} = 100 - 73 = 27$, so $c = 9$.

The height drawn from C to AB is

$$m = \sqrt{BC^2 - \left(\frac{c}{2}\right)^2} = \sqrt{9 \cdot 18.25 - 20.25} = \sqrt{144} = 12$$

and the area is $\frac{9 \cdot 12}{2} = 54$ square units.

6. The angles α and β of a triangle satisfy

$$(1 + \tan \alpha)(1 + \tan \beta) = 2$$

Find the third angle of the triangle.

Solution: (Since $\tan \alpha$ and $\tan \beta$ appear, neither angle is 90° .) Multiplying out,

$$1 + \tan \alpha + \tan \beta + \tan \alpha \tan \beta = 2 \quad \text{so} \quad \tan \alpha + \tan \beta = 1 - \tan \alpha \tan \beta$$

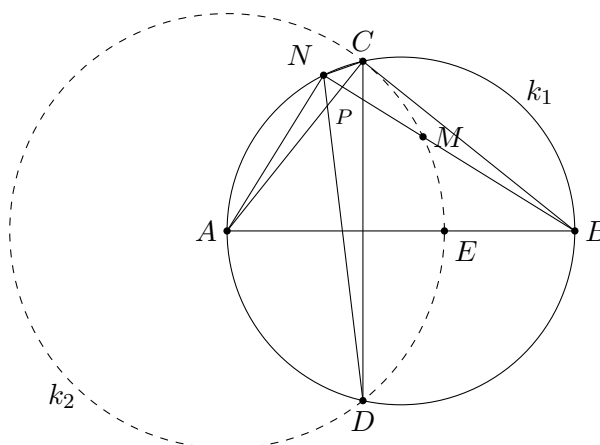
If $1 - \tan \alpha \tan \beta = 0$, then $\tan \alpha + \tan \beta = 0$ too, so $\tan \beta = -\tan \alpha$ and $\tan \alpha \tan \beta = -\tan^2 \alpha \leq 0$, which contradicts $\tan \alpha \tan \beta = 1$. So we can divide:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = 1$$

Since $0^\circ < \alpha + \beta < 180^\circ$, we get $\alpha + \beta = 45^\circ$, and the third angle is $180^\circ - 45^\circ = 135^\circ$.

7. The circle k_2 with center A intersects the circle k_1 with diameter AB at points C and D . Let E be the point where k_2 intersects the segment AB . Let M be any point inside the arc CE of k_2 that is inside triangle ABC , and let N be the second intersection of the line BM with k_1 . Prove that $MN^2 = CN \cdot DN$.

Solution:



Step 1. By Thales' theorem, $\angle ACB = 90^\circ$, so BC is perpendicular to the radius AC of k_2 : BC is tangent to k_2 . Let M' be the second intersection of line BM with k_2 . Comparing the tangent and the secant from B , $BC^2 = BM \cdot BM'$.

Step 2. By Thales' theorem again, $\angle ANB = 90^\circ$, so AN is perpendicular to the chord MM' of k_2 . The perpendicular from the center bisects the chord, so N is the midpoint of MM' . Then $BM = BN - MN$ and $BM' = BN + MN$, so

$$BC^2 = (BN - MN)(BN + MN) = BN^2 - MN^2 \quad \text{and} \quad MN^2 = BN^2 - BC^2$$

Step 3. The common chord CD is perpendicular to the line of the centers, AB , so $BC = BD$ and the arcs BC and BD of k_1 are equal. Let P be the intersection of BN and CD . Since equal arcs belong to equal inscribed angles, $\angle BNC = \angle BCD$, so triangles BCP and BNC are similar (they share the angle at B). So $\frac{BC}{BN} = \frac{BP}{BC}$, and

$$BN^2 - BC^2 = BN^2 - BN \cdot BP = BN \cdot PN$$

Also, $\angle CNP = \angle BND$ (equal arcs) and $\angle NCP = \angle NCD = \angle NBD$ (both subtend the arc ND), so triangles NCP and NBD are similar, and $\frac{NC}{NB} = \frac{NP}{ND}$, that is, $CN \cdot DN = BN \cdot PN$.

Putting the steps together, $MN^2 = BN^2 - BC^2 = BN \cdot PN = CN \cdot DN$.

8. From a four-digit number, we subtract the numbers we get by deleting its last digit, its last two digits, and its last three digits. The result is 1765. Find the number.

Solution: Let the number be $\overline{abcd} = 1000a + 100b + 10c + d$. The numbers we subtract are $\overline{abc} = 100a + 10b + c$, $\overline{ab} = 10a + b$, and a . So

$$1000a + 100b + 10c + d - (100a + 10b + c) - (10a + b) - a = 889a + 89b + 9c + d = 1765$$

If $a \geq 2$, the left-hand side is at least 1778, so $a = 1$, and $89b + 9c + d = 876$. Since $9c + d \leq 90$, we need $89b \geq 786$, so $b = 9$ and $9c + d = 75$. Since $d \leq 9$, $c = 8$ and $d = 3$.

The number is 1983. (Check: $1983 - 198 - 19 - 1 = 1765$.)