

author: Imre Rábai

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The bisector of the right angle of a right triangle divides the hypotenuse in a 1 : 3 ratio. In what ratio does the altitude belonging to the hypotenuse divide the hypotenuse?

Solution: Let the legs be  $a$  and  $b$ . By the angle bisector theorem, the bisector divides the hypotenuse in the ratio of the legs, so  $a : b = 1 : 3$ .

The altitude divides the hypotenuse  $c$  into two segments  $p$  and  $q$ , where  $p$  is next to leg  $a$  and  $q$  is next to leg  $b$ . By the leg theorem,  $a^2 = pc$  and  $b^2 = qc$ , so

$$\frac{p}{q} = \frac{a^2}{b^2} = \frac{1}{9}$$

The altitude divides the hypotenuse in a 1 : 9 ratio.

2. Solve each of the following equations.

$$\text{a) } 2x - 3 = |3 - 2x| \quad \text{b) } \sqrt[3]{2^{2x-3}} = 8^{x-\frac{3}{2}} \quad \text{c) } \log_{2x-3} \sqrt{2x-3} = \frac{1}{2}$$

Solution: a) Since  $|3 - 2x| = |2x - 3|$ , the equation says that  $2x - 3$  equals its own absolute value. This is true exactly when  $2x - 3 \geq 0$ , so the solutions are all  $x \geq \frac{3}{2}$ .

b) We write both sides as powers of 2:  $2^{\frac{2x-3}{3}} = 2^{3x-\frac{9}{2}}$ . So  $\frac{2x-3}{3} = 3x - \frac{9}{2}$ , that is,  $4x - 6 = 18x - 27$ , and  $x = \frac{3}{2}$ .

c) The base must be positive and not 1:  $2x - 3 > 0$  and  $2x - 3 \neq 1$ , that is,  $x > \frac{3}{2}$  and  $x \neq 2$ . For every such  $x$ ,  $\log_{2x-3} \sqrt{2x-3} = \frac{1}{2}$  (since  $\sqrt{2x-3} = (2x-3)^{\frac{1}{2}}$ ). So the solutions are all  $x > \frac{3}{2}$  with  $x \neq 2$ .

3. The height of a frustum of a pyramid with a triangular base is 10 units, and the edges of its base are 27, 29, and 52 units long. The perimeter of the top face is 72 units. Find the volume of the frustum.

Solution: The top face is similar to the base, and the ratio of similarity is the ratio of the perimeters:  $\frac{72}{27 + 29 + 52} = \frac{72}{108} = \frac{2}{3}$ . The area of the base, by Heron's formula with  $s = 54$ :

$$T = \sqrt{54 \cdot 27 \cdot 25 \cdot 2} = \sqrt{72900} = 270$$

The ratio of the areas is the square of the ratio of similarity, so the area of the top face is  $t = 270 \cdot \frac{4}{9} = 120$ . The volume of the frustum is

$$V = \frac{h}{3} (T + \sqrt{Tt} + t) = \frac{10}{3} (270 + \sqrt{32400} + 120) = \frac{10}{3} (270 + 180 + 120) = 1900$$

cubic units.

4. The endpoints of one of the diagonals of rhombus  $ABCD$  are  $A(-4, -5)$  and  $C(5, 4)$ , and the area of the rhombus is 27 square units. Find the coordinates of the other two vertices.

Solution: The length of diagonal  $AC$  is  $\sqrt{9^2 + 9^2} = 9\sqrt{2}$ . The area of a rhombus is half of the product of its diagonals, so

$$BD = \frac{2 \cdot 27}{9\sqrt{2}} = 3\sqrt{2}$$

The diagonals bisect each other at the midpoint  $M\left(\frac{1}{2}, -\frac{1}{2}\right)$  of  $AC$ , and they are perpendicular. The direction of  $AC$  is  $(1, 1)$ , so the direction of  $BD$  is  $(-1, 1)$ , with length  $\sqrt{2}$ . So

$$B, D = \left(\frac{1}{2}, -\frac{1}{2}\right) \pm \frac{3\sqrt{2}}{2} \cdot \frac{(-1, 1)}{\sqrt{2}} = \left(\frac{1}{2}, -\frac{1}{2}\right) \pm \frac{3}{2}(-1, 1)$$

The other two vertices are  $(-1, 1)$  and  $(2, -2)$ .

5. Solve the following systems of equations.

a)  $\tan x + \tan y = 2\sqrt{3}, \quad \tan x \cdot \tan y = 3$

b)  $x + y = \frac{2\pi}{3}, \quad \tan x \cdot \tan y = 3$

Solution: a) By Vieta's formulas,  $\tan x$  and  $\tan y$  are the roots of  $t^2 - 2\sqrt{3}t + 3 = (t - \sqrt{3})^2 = 0$ . So  $\tan x = \tan y = \sqrt{3}$ , and the solutions are  $x = \frac{\pi}{3} + k\pi, y = \frac{\pi}{3} + n\pi$ , where  $k$  and  $n$  are any integers.

b) By the addition formula,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \quad \text{so} \quad -\sqrt{3} = \frac{\tan x + \tan y}{1 - 3}$$

and  $\tan x + \tan y = 2\sqrt{3}$ . So by part a),  $\tan x = \tan y = \sqrt{3}$ , and with  $x + y = \frac{2\pi}{3}$ , the solutions are

$$x = \frac{\pi}{3} + k\pi, \quad y = \frac{\pi}{3} - k\pi$$

where  $k$  is any integer.

6. The sides of a triangle are  $a, b, c$ , and the opposite angles are  $\alpha, \beta, \gamma$ , respectively.

a) Find the angles of the triangle if  $2(a + c) = 3b$  and  $\gamma - \beta = 60^\circ$ .

b) Find the sides of this triangle if the radius of its circumscribed circle is 5 units.

Solution: a) By the Law of Sines,  $a : b : c = \sin \alpha : \sin \beta : \sin \gamma$ , so the condition is  $2(\sin \alpha + \sin \gamma) = 3 \sin \beta$ . We use the sum-to-product identity and  $\frac{\alpha + \gamma}{2} = 90^\circ - \frac{\beta}{2}$ :

$$\sin \alpha + \sin \gamma = 2 \sin \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} = 2 \cos \frac{\beta}{2} \cos \frac{\alpha - \gamma}{2}$$

and  $\sin \beta = 2 \sin \frac{\beta}{2} \cos \frac{\beta}{2}$ . Dividing by  $4 \cos \frac{\beta}{2}$ , the condition becomes  $2 \cos \frac{\alpha - \gamma}{2} = 3 \sin \frac{\beta}{2}$ .

Since  $\gamma = \beta + 60^\circ$ , we have  $\alpha = 120^\circ - 2\beta$  and  $\frac{\alpha - \gamma}{2} = 30^\circ - \frac{3\beta}{2}$ . Let  $u = \frac{\beta}{2}$ :

$$\begin{aligned} 2 \cos (30^\circ - 3u) &= 3 \sin u \\ \sqrt{3} \cos 3u + \sin 3u &= 3 \sin u \quad \sin 3u = 3 \sin u - 4 \sin^3 u \\ \sqrt{3} \cos 3u &= 4 \sin^3 u \quad \cos 3u = 4 \cos^3 u - 3 \cos u \end{aligned}$$

Dividing by  $\cos^3 u$  and writing  $t = \tan u$  (so  $\frac{1}{\cos^2 u} = 1 + t^2$ ),

$$\sqrt{3}(4 - 3(1 + t^2)) = 4t^3 \quad \text{or} \quad 4t^3 + 3\sqrt{3}t^2 - \sqrt{3} = 0$$

This cubic equation has only one positive root,  $t \approx 0.4917$ , and it has no nice exact value. So  $\frac{\beta}{2} \approx 26.18^\circ$ , and the angles are

$$\beta \approx 52.37^\circ, \quad \gamma = \beta + 60^\circ \approx 112.37^\circ, \quad \alpha = 120^\circ - 2\beta \approx 15.26^\circ$$

(Remark: the numbers are not nice, so the original problem may contain a misprint. For example, with the condition  $\gamma - \alpha = 60^\circ$  instead, we would get  $\cos \frac{\alpha - \gamma}{2} = \frac{\sqrt{3}}{2}$ ,  $\sin \frac{\beta}{2} = \frac{1}{\sqrt{3}}$ , and  $\cos \beta = \frac{1}{3}$ .)

b) By the Law of Sines, each side is  $2R = 10$  times the sine of the opposite angle:

$$a = 10 \sin \alpha \approx 2.63, \quad b = 10 \sin \beta \approx 7.92, \quad c = 10 \sin \gamma \approx 9.25$$

(Check:  $2(a + c) \approx 23.76 \approx 3b$ .)

7. Find the values of the parameter  $a$  for which the system of equations

$$\begin{aligned} x + y &= -2 - xy \\ x - y &= a + axy \end{aligned}$$

has exactly one solution.

Solution: The first equation is  $xy + x + y + 1 = -1$ , that is,  $(x + 1)(y + 1) = -1$ . Let  $u = x + 1$  and  $v = y + 1$ ; then  $uv = -1$ , so  $u \neq 0$  and  $v = -\frac{1}{u}$ . Also  $x - y = u - v$  and  $xy = (u - 1)(v - 1) = uv - u - v + 1 = -(u + v)$ , so the second equation becomes

$$u - v = a - a(u + v) \quad \text{that is,} \quad (1 + a)u - (1 - a)v = a$$

Substituting  $v = -\frac{1}{u}$  and multiplying by  $u$ :

$$(1+a)u^2 - au + (1-a) = 0, \quad u \neq 0$$

Each nonzero root  $u$  gives exactly one solution of the system, so we need exactly one nonzero root.

If  $a = -1$ : the equation is  $u + 2 = 0$ , with the one root  $u = -2$ . (Solution:  $x = -3, y = -\frac{1}{2}$ .)

If  $a = 1$ : the equation is  $2u^2 - u = 0$ , with roots 0 and  $\frac{1}{2}$ ; only  $u = \frac{1}{2}$  is allowed. (Solution:  $x = -\frac{1}{2}, y = -3$ .)

Otherwise  $u = 0$  is not a root, and we need a double root: the discriminant  $a^2 - 4(1+a)(1-a) = 5a^2 - 4$  must be 0, so  $a = \pm \frac{2}{\sqrt{5}}$ .

So the system has exactly one solution if and only if  $a = \pm 1$  or  $a = \pm \frac{2\sqrt{5}}{5}$ .

8. Find the largest possible value of  $ab$  if

$$0 < a < 1, \quad 0 < b < 1, \quad \text{and} \quad \log_{\frac{1}{2}} a \cdot \log_{\frac{1}{2}} b = 1$$

Solution: Let  $p = \log_{\frac{1}{2}} a$  and  $q = \log_{\frac{1}{2}} b$ . Since  $0 < a < 1$  and  $0 < b < 1$ , both  $p$  and  $q$  are positive, and  $pq = 1$ .

Also  $a = \left(\frac{1}{2}\right)^p$  and  $b = \left(\frac{1}{2}\right)^q$ , so

$$ab = \left(\frac{1}{2}\right)^{p+q}$$

This is largest when  $p+q$  is smallest. By the inequality between the arithmetic and geometric means,  $p+q \geq 2\sqrt{pq} = 2$ , with equality if and only if  $p = q = 1$ . So the largest value of  $ab$  is  $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ , attained when  $a = b = \frac{1}{2}$ .