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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. We have used an entire record tape to record sounds. On one side of the tape, we have recorded sounds with a speed of $4\frac{\text{cm}}{\text{s}}$. On the other side of the tape, we have recorded sound with a speed of $9\frac{\text{cm}}{\text{s}}$. The length of the entire program we have taped is 1 hour and 18 minutes. How many meters long is the ribbon in the tape? How many minutes can we tape if we record with a speed of $9\frac{\text{cm}}{\text{s}}$ on both sides?

   Solution: Let L denote the length of the ribbon of the tape in centimeters.
   $t_1 9\frac{\text{cm}}{\text{s}} = L$ and $t_2 4\frac{\text{cm}}{\text{s}} = L$
   1 hour 18 minutes$=78$ minutes $=4680$ seconds

$$\begin{aligned} t_1 + t_2 &= \frac{L}{9} + \frac{L}{4} = 4680 \\ L\left(\frac{1}{9} + \frac{1}{4}\right) &= 4680 \\ L &= \frac{4680\,\text{s}}{\frac{1}{9\frac{\text{cm}}{\text{s}}} + \frac{1}{4\frac{\text{cm}}{\text{s}}}} = \frac{4680\,\text{s}}{\frac{4+9}{36}\frac{\text{s}}{\text{cm}}} = 4680\,\text{s} \cdot \frac{36}{13}\frac{\text{cm}}{\text{s}} = 12\,960\,\text{cm} = 12.9\,\text{m} \end{aligned}$$

   So $L = 12960\,\text{cm} = 12.9$ meters. And if we tape with a speed of $9\frac{\text{cm}}{\text{s}}$ on both sides, we have

$$t = \frac{L}{9\frac{\text{cm}}{\text{s}}} + \frac{L}{9\frac{\text{cm}}{\text{s}}} = 2 \cdot \frac{12960\,\text{cm}}{9\frac{\text{cm}}{\text{s}}} = 2880\,\text{s} = 48\,\text{min}$$

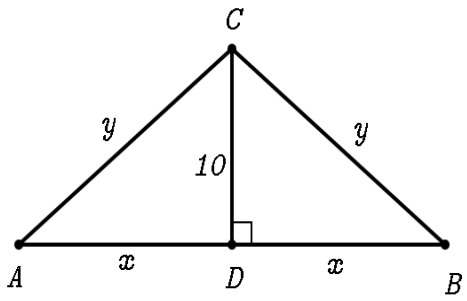
   so we can tape 48 minutes.

2. In an isosceles triangle, the altitude (or height) belonging to the base is 10 units long. The altitude belonging to the other two sides are both 12 units long. Find the area of the triangle.

Solution: Let x denote half of the base. Then the area of the triangle is $A = \frac{1}{2}(2x)\,10 = 10x$. Let y denote the other side. The area of the triangle is $A = \frac{1}{2}y\,(12) = 6y$. The area must be the same, no matter which side we used to compute it. So,

$$\begin{aligned} 6y &= 10x \\ y &= \frac{5}{3}x \end{aligned}$$

Consider now the right triangle ADC as shown on the picture.



We state the Pythagorean Theorem and recall that $y = \frac{5}{3}x$

$$\begin{aligned} x^2 + 10^2 &= \left(\frac{5}{3}x\right)^2 \\ x^2 + 100 &= \frac{25}{9}x^2 \\ 100 &= \frac{16}{9}x^2 \\ \frac{900}{16} &= x^2 \\ x &= \pm\frac{30}{4} = \pm\frac{15}{2} \end{aligned}$$

Since x denotes a distance, the negative solution is easily discarded.

$$x = \frac{15}{2}$$

Then the area of the triangle is $A = 10x = 10 \cdot \frac{15}{2} = 75$ unit^2.

3. Solve the given equation over the real numbers.

$$\frac{1}{\sqrt{2+x}-\sqrt{2-x}}+\frac{1}{\sqrt{2+x}+\sqrt{2-x}}=1$$

Solution: before we start solving the equation, notice the domain. We will have to worry about

1) $2+x\geq 0$ thus $x\geq -2$

2) $2-x\geq 0$ thus $x\leq 2$ and so $-2\leq x\leq 2$

But also $\sqrt{2+x}-\sqrt{2-x}=0$ and $\sqrt{2+x}+\sqrt{2-x}=0$ must be ruled out. We can check our result against the last two at the end. We bring the fractions to the common denominator:

$$\begin{aligned}
\frac{1}{\sqrt{2+x}-\sqrt{2-x}}+\frac{1}{\sqrt{2+x}+\sqrt{2-x}} &= 1\\
\frac{\sqrt{2+x}+\sqrt{2-x}+\left(\sqrt{2+x}-\sqrt{2-x}\right)}{\left(\sqrt{2+x}-\sqrt{2-x}\right)\left(\sqrt{2+x}+\sqrt{2-x}\right)} &= 1\\
\frac{\sqrt{2+x}+\sqrt{2-x}+\sqrt{2+x}-\sqrt{2-x}}{\left(\sqrt{2+x}\right)^2-\left(\sqrt{2-x}\right)^2} &= 1\\
\frac{2\sqrt{2+x}}{(2+x)-(2-x)} &= 1\\
\frac{2\sqrt{2+x}}{2x} &= 1\\
\frac{\sqrt{2+x}}{x} &= 1 \quad \text{multiply by } x\\
\sqrt{2+x} &= x \quad \text{square}\\
2+x &= x^2\\
0 &= x^2-x-2\\
0 &= (x-2)(x+1)
\end{aligned}$$

$$x_1=2,\quad x_2=-1$$

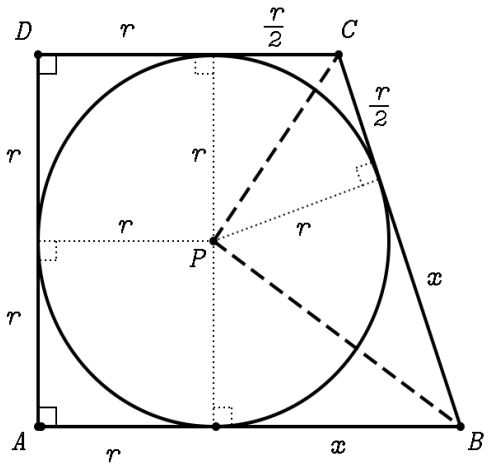
Since x must be between -2 and 2, $x_2=-1$ is ruled out. We check for $x=2$:

$$\text{LHS}=\frac{1}{\sqrt{2+2}-\sqrt{2-2}}+\frac{1}{\sqrt{2+2}+\sqrt{2-2}}=\frac{1}{\sqrt{4}-\sqrt{0}}+\frac{1}{\sqrt{4}+\sqrt{0}}=\frac{1}{2-0}+\frac{1}{2+0}=\frac{1}{2}+\frac{1}{2}=1=\text{RHS}$$

and so our solution is correct.

4. Suppose that C is a circle with radius r. We draw a trapezoid around the circle so that each side of the trapezoid is tangent to the circle. One of the angles of the trapezoid is $90°$. The shortest side of the trapezoid is $\frac{3r}{2}$. Compute the area and perimeter of the trapezoid.

Solution: Recall that the tangential segments drawn from an outer point are equally long. Consider the picture shown.



First we will show that the angle $CPB = 90°$. Since sides DC and AB are parallel, we have that $\angle ABC + \angle BCD = 180°$. CP bisects angle BCD and BP bisects angle ABC. Thus

$$\angle PBC + \angle BCP = \frac{\angle ABC}{2} + \frac{\angle BCD}{2} = \frac{180°}{2} = 90°$$

Then $\angle CPB = 90°$.

Second, the altitude drawn to the hypotenuse is the geometric mean of the two line segments it splits the hypotenuse into: $r^2 = \frac{r}{2} \cdot x$ and so $x = 2r$. The rest is easy.

The area is

$$A = \frac{AB + CD}{2} \cdot AD = \frac{(r + x) + \left(\frac{3}{2}r\right)}{2} \cdot 2r = \frac{(r + 2r) + \left(\frac{3}{2}r\right)}{2} \cdot 2r = \left(3r + \frac{3}{2}r\right) r = \frac{9}{2}r^2$$

and the perimeter is

$$\begin{aligned} P &= AB + BC + CD + DA = (r + x) + \left(x + \frac{r}{2}\right) + \frac{3}{2}r + 2r = (r + 2r) + \left(2r + \frac{r}{2}\right) + \frac{3}{2}r + 2r \\ &= 3r + \frac{5}{2}r + \frac{3}{2}r + 2r = 9r \end{aligned}$$

5. Write an equation for the BC side side in triangle ABC, where the equation of side AB is $3x-2y+1=0$, the equation for side AC is $x-y+1=0$, and the median that contains C has equation $2x-y-1=0$.
(A median is a line segment connecting the midpoint of a side with the vertex opposite that side).

Solution: For vertex A, we just find the intersection point of AB and AC.

$$\begin{aligned} 3x-2y+1 &= 0 \\ x-y+1 &= 0 \quad \Longrightarrow \quad x=y-1 \end{aligned}$$

$$\begin{aligned} 3x-2y+1 &= 0 \\ 3(y-1)-2y+1 &= 0 \\ 3y-3-2y+1 &= 0 \\ y-2 &= 0 \\ y &= 2 \quad \Longrightarrow \quad x=2-1=1 \qquad \text{So } A(1,2) \end{aligned}$$

For vertex C, we intersect side AC with the median given:

$$\begin{aligned} x-y+1 &= 0 \quad \Longrightarrow \quad x=y-1 \\ 2x-y-1 &= 0 \end{aligned}$$

$$\begin{aligned} 2(y-1)-y-1 &= 0 \\ 2y-2-y-1 &= 0 \\ y-3 &= 0 \\ y &= 3 \quad \Longrightarrow \quad x=3-1=2 \qquad \text{So } C(2,3) \end{aligned}$$

Let M be the midpoint of side AB. We can find M by intersecting the median with side AB.

$$\begin{aligned} 3x-2y+1 &= 0 \\ 2x-y-1 &= 0 \quad \Longrightarrow \quad y=2x-1 \end{aligned}$$

$$\begin{aligned} 3x-2y+1 &= 0 \\ 3x-2(2x-1)+1 &= 0 \\ 3x-4x+2+1 &= 0 \\ -x+3 &= 0 \\ 3 &= x \quad \Longrightarrow \quad y=2\cdot 3-1=5 \qquad \text{So } M(3,5) \end{aligned}$$

We can now find the coordinates of B, because M is the midpoint of AB.

$$\begin{aligned} x_M &= \frac{x_A+x_B}{2} \text{ and } y_M=\frac{y_A+y_B}{2} \\ 3 &= \frac{1+x_B}{2} \text{ and } 5=\frac{2+y_B}{2} \\ 6 &= 1+x_B \text{ and } 10=2+y_B \\ 5 &= x_B \text{ and } 8=y_B \qquad \text{So } B(5,8) \end{aligned}$$

We now have to find the equation of the line connecting $B(5,8)$ and $C(2,3)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{8-3}{5-2}=\frac{5}{3}$$

the point-slope form ofthe equation is

$$\begin{aligned}\frac{5}{3}(x-2) &= y-3\\ 5(x-2) &= 3(y-3)\\ 5x-10 &= 3y-9\\ 5x-3y &= 1\end{aligned}$$

6. Prove that if α, β, and γ are angles in a triangle, then

$$\sin^2\gamma \geq \sin 2\alpha \cdot \sin 2\beta$$

For what triangles is the equality true?

Solution: $\gamma = 180^\circ - (\alpha+\beta)$ and $\sin\gamma = \sin(180^\circ-(\alpha+\beta)) = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\begin{aligned}\sin^2\gamma &\geq \sin 2\alpha\cdot\sin 2\beta\\ (\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2 &\geq 2\sin\alpha\cos\alpha\cdot 2\sin\beta\cos\beta\\ \sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+2\sin\alpha\cos\alpha\sin\beta\cos\beta &\geq 4\sin\alpha\cos\alpha\sin\beta\cos\beta\\ \sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta-2\sin\alpha\cos\alpha\sin\beta\cos\beta &\geq 0\\ (\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2 &\geq 0\end{aligned}$$

Since the left-hand side is a square, it is clearly non-negative. For the equality, notice that $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\beta-\alpha)$

$$\begin{aligned}(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2 &= 0\\ \sin(\beta-\alpha) &= 0\\ \beta-\alpha &= k\cdot 180^\circ \quad \text{where } k\in\mathbb{Z}\\ \beta &= \alpha + k\cdot 180^\circ\end{aligned}$$

Since both angles are between 0 and 180°, $k=0$ and so $\alpha=\beta$ and so the triangle is isosceles.

7. A rectangular prism has a square base. The ratio of its height and side of its base is an integer. The midpoints of the four sides of the base and the midpoints of the four sides of the top form another rectangular prism. The ratio between the surface area of the second prism and six times the area of the base of the first prism is between 0.5 and 2. How many different such rectanguar prism exist if we consider similar prisms as identical?

Solution: Let x denote the side of the base and nx the vertical side, where n is a positive integer.

The second prism has a base with sides $\frac{x}{\sqrt{2}}$. So the surface area of the second prism is

$$S_2 = 2\left(\frac{x}{\sqrt{2}}\right)^2 + 4\left(\frac{x}{\sqrt{2}}(nx)\right) = x^2 + \frac{4}{\sqrt{2}}nx^2 = \left(1+2\sqrt{2}n\right)x^2$$

The other area:

$$A = 6x^2$$

So

$$\begin{aligned}
0.5 &\leq \frac{S_2}{A} \leq 2 \\
0.5 &\leq \frac{\left(1+2\sqrt{2}n\right)x^2}{6x^2} \leq 2 \\
0.5 &\leq \frac{1+2\sqrt{2}n}{6} \leq 2 \\
3 &\leq 1+2\sqrt{2}n \leq 12 \\
2 &\leq 2\sqrt{2}n \leq 11 \\
\frac{2}{2\sqrt{2}} &\leq n \leq \frac{11}{2\sqrt{2}} \\
\frac{\sqrt{2}}{2} &\leq n \leq \frac{11\sqrt{2}}{4} \\
\sqrt{\frac{1}{2}} &\leq n \leq \sqrt{\frac{242}{16}}
\end{aligned}$$

Since $\sqrt{\frac{1}{2}} \approx 0.707\,107$ and $\sqrt{\frac{242}{16}} = 3.889$ which means that

$$n = 1, 2, 3$$

and so there are three different such prisms.

8. Suppose that p is a positive integer. If we add the digits of p, the result is q. If we add the digits of q, the result is r. Find p if we also know that $p + q + r = 60$.

Solution: Recall that when we add the digits of a number, the sum and the original number have the same remainder when divided by 9. Clearly p, q, $r < 60$. Let's think about this with respect to divisibility by 9. $p \equiv q \equiv r \pmod 9$ and so

$$\begin{aligned}
p+q+r &\equiv 3p \equiv 60 \equiv 6 \mod 9 \\
3p &\equiv 6 \mod 9 \\
p &\equiv 2 \mod 9
\end{aligned}$$

So $p \in \{2, 11, 20, 29, 38, 47, 56\}$. We check these: If $p = 2$ then $q = r = 2$, then $p + q + r = 6$ no good.

If $p = 11$, then $q = 2$ and $r = 2$. $p + q + r = 11 + 2 + 2 = 15$ no good.

If $p = 20$, then $q = 2$ and $r = 2$. $p + q + r = 6$ no good.

If $p = 29$, then $q = 11$ and $r = 2$. $p + q + r = 29 + 11 + 2 = 42$ no good.

If $p = 38$, then $q = 11$ and $r = 2$. $p + q + r = 51$ no good.

If $p = 47$, then $q = 11$ and $r = 2$. $p + q + r = 60$ this one works.

If $p = 56$, then $q = 11$ and $r = 2$. $p + q + r = 69$ no good. So $p = 47$.