

author: János Rácz

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the circles $K_1: (x - 5)^2 + (y + 2)^2 - 17 = 0$ and $K_2: (x + 1)^2 + (y - 5)^2 - 68 = 0$. Let C be the center of K_1 , D the center of K_2 , and let A and B be the intersection points of the two circles. Find the area of quadrilateral $CADB$.

Solution: The center of K_1 is $C(5, -2)$, its radius is $\sqrt{17}$; the center of K_2 is $D(-1, 5)$, its radius is $\sqrt{68}$. The distance of the centers is $CD = \sqrt{6^2 + 7^2} = \sqrt{85}$.

In triangle CAD , $CA = \sqrt{17}$, $DA = \sqrt{68}$, and $CD = \sqrt{85}$. Since $17 + 68 = 85$, by the converse of the Pythagorean theorem, the angle at A is a right angle. The same is true for triangle CBD , and the two triangles are congruent (they are reflections of each other about CD). So the area of the quadrilateral is

$$T = 2 \cdot \frac{\sqrt{17} \cdot \sqrt{68}}{2} = \sqrt{17 \cdot 68} = \sqrt{1156} = 34 \text{ unit}^2$$

2. For which values of the real parameter k does the quadratic equation

$$(k + 2)x^2 - 2(k - 11)x + (k - 3)(k + 5) = 0$$

have two real roots with opposite signs?

Solution: For a quadratic equation we need $k \neq -2$. The roots are real and have opposite signs if and only if their product is negative. (If the product $\frac{c}{a}$ is negative, then $ac < 0$, so the discriminant $b^2 - 4ac$ is positive and the roots are real.) By Vieta's formulas, we need

$$\frac{(k - 3)(k + 5)}{k + 2} < 0$$

The expression changes sign at -5 , -2 , and 3 . Checking the signs on the intervals, it is negative for $k < -5$ and for $-2 < k < 3$. So the answer is $k < -5$ or $-2 < k < 3$.

3. Solve the given equation.

$$\sqrt{x+20} - \sqrt{x-15} = x-9$$

Solution: The domain is $x \geq 15$. Rationalizing the left-hand side,

$$\sqrt{x+20} - \sqrt{x-15} = \frac{(x+20) - (x-15)}{\sqrt{x+20} + \sqrt{x-15}} = \frac{35}{\sqrt{x+20} + \sqrt{x-15}}$$

This is decreasing on the domain, and its largest value (at $x = 15$) is $\frac{35}{\sqrt{35}} = \sqrt{35} < 6$. The right-hand side $x - 9$ is increasing, and its smallest value is $15 - 9 = 6$. So the left-hand side is always less than the right-hand side, and the equation has no solution.

4. Consider the circle $K: (x+3)^2 + (y-2)^2 - 19 = 0$ and the line $L: 11x - 2y + 16 = 0$. Find the point P for which the line connecting the points of tangency of the two tangent lines drawn from P to the circle K is the given line L .

Solution: The center of K is $O(-3, 2)$ and its radius is $r = \sqrt{19}$. Let T be one of the points of tangency. Since $OT \perp TP$, triangle OTP has a right angle at T , and the chord of the points of tangency is perpendicular to OP . So P is on the line through O perpendicular to L , and if M is the intersection of this line with L , then TM is the height of the right triangle OTP . By the leg theorem,

$$OT^2 = OM \cdot OP, \quad \text{that is,} \quad OP = \frac{19}{OM}$$

A normal vector of L is $(11, -2)$, so the points of the perpendicular are $O + t(11, -2) = (-3 + 11t, 2 - 2t)$. For M :

$$11(-3 + 11t) - 2(2 - 2t) + 16 = 0 \quad \text{so} \quad 125t = 21 \quad \text{and} \quad t_M = \frac{21}{125}$$

Since $OM = t_M \sqrt{125}$ and $OP = t_P \sqrt{125}$, we need $t_P \cdot t_M \cdot 125 = 19$, so $t_P = \frac{19}{21}$, and

$$P = \left(-3 + \frac{209}{21}, 2 - \frac{38}{21} \right) = \left(\frac{146}{21}, \frac{4}{21} \right)$$

(The line L intersects the circle, since $OM = \frac{21}{\sqrt{125}} \approx 1.88 < \sqrt{19}$, so the tangent lines exist.)

5. Solve the given equation.

$$2 \cos \frac{x}{2} + \sqrt{\cos 2x - 3 \cos x} = 0$$

Solution: The square root is non-negative, so we need $\cos \frac{x}{2} \leq 0$ and $\sqrt{\cos 2x - 3 \cos x} = -2 \cos \frac{x}{2}$. Squaring, and using $4 \cos^2 \frac{x}{2} = 2(1 + \cos x)$ and $\cos 2x = 2 \cos^2 x - 1$:

$$2 \cos^2 x - 1 - 3 \cos x = 2 + 2 \cos x$$

$$2 \cos^2 x - 5 \cos x - 3 = 0$$

$$(2 \cos x + 1)(\cos x - 3) = 0$$

So $\cos x = -\frac{1}{2}$. Then $\cos 2x - 3 \cos x = -\frac{1}{2} + \frac{3}{2} = 1$, and the equation becomes $2 \cos \frac{x}{2} + 1 = 0$, that is, $\cos \frac{x}{2} = -\frac{1}{2}$.

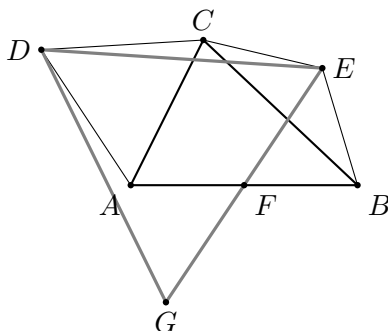
So $\frac{x}{2} = \pm \frac{2\pi}{3} + 2k\pi$, and

$$x = \pm \frac{4\pi}{3} + 4k\pi \quad \text{where } k \text{ is any integer.}$$

(These values indeed satisfy $\cos x = -\frac{1}{2}$.)

6. Consider a triangle ABC . Let D be the third vertex of the equilateral triangle drawn outward on side AC , and let E be the vertex with the 120° angle of the isosceles triangle with a 120° angle drawn outward on side BC . Let F be the midpoint of side AB , and let G be the reflection of E about F . What kind of triangle is DEG ?

Solution: Triangle DEG is equilateral.



Notice that E is the center of the equilateral triangle drawn outward on BC , so the rotation about E by 120° takes B to C . Also, the rotation about D by 60° takes C to A . Since both triangles were drawn outward, these two rotations turn in the same direction.

Apply the first rotation and then the second one. The result is a rotation by $120^\circ + 60^\circ = 180^\circ$, that is, a reflection about a point, and it takes B to C and then to A . A reflection about a point that takes B to A must be the reflection about the midpoint F of AB .

Now apply this to the point E . The first rotation does not move E (it is the center), and the second one takes E to some point E' . On the other hand, the reflection about F takes E to G . So $G = E'$, that is, the rotation about D by 60° takes E to G . Therefore $DE = DG$ and $\angle EDG = 60^\circ$, and so triangle DEG is equilateral.

7. Solve the given inequality.

$$2 \cos x \sin \frac{x}{2} + \sqrt{1 - \cos x} \geq 0$$

Solution: Since $1 - \cos x = 2 \sin^2 \frac{x}{2}$, we have $\sqrt{1 - \cos x} = \sqrt{2} \left| \sin \frac{x}{2} \right|$. Let $s = \sin \frac{x}{2}$. The inequality is

$$2s \cos x + \sqrt{2} |s| \geq 0$$

The left-hand side has period 4π , so we solve it on $[0, 4\pi)$.

If $s = 0$ ($x = 0$ or $x = 2\pi$), the inequality holds.

If $s > 0$, that is, $0 < x < 2\pi$: dividing by s , we need $\cos x \geq -\frac{\sqrt{2}}{2}$, so $0 < x \leq \frac{3\pi}{4}$ or $\frac{5\pi}{4} \leq x < 2\pi$.

If $s < 0$, that is, $2\pi < x < 4\pi$: then $|s| = -s$, and dividing by $s < 0$ reverses the inequality: $2 \cos x - \sqrt{2} \leq 0$, so $\cos x \leq \frac{\sqrt{2}}{2}$, that is, $\frac{9\pi}{4} \leq x \leq \frac{15\pi}{4}$.

So the solution is

$$4k\pi \leq x \leq \frac{3\pi}{4} + 4k\pi, \quad \frac{5\pi}{4} + 4k\pi \leq x \leq 2\pi + 4k\pi, \quad \text{or} \quad \frac{9\pi}{4} + 4k\pi \leq x \leq \frac{15\pi}{4} + 4k\pi$$

where k is any integer.

8. Solve the given inequality,

$$\log_a b + \log_b a \leq 2, \quad \text{where} \quad a = \frac{x^2 - x + 20}{50} \quad \text{and} \quad b = \frac{x^2 - 8x + 17}{50}$$

Solution: Both $x^2 - x + 20$ and $x^2 - 8x + 17 = (x - 4)^2 + 1$ are always positive, so $a > 0$ and $b > 0$. We also need $a \neq 1$ and $b \neq 1$:

$$x^2 - x - 30 = (x - 6)(x + 5) \neq 0, \quad x^2 - 8x - 33 = (x - 11)(x + 3) \neq 0$$

so $x \neq 6, -5, 11, -3$. Let $u = \log_a b$. Then $u \neq 0$ and $\log_b a = \frac{1}{u}$, so

$$u + \frac{1}{u} \leq 2 \quad \Longleftrightarrow \quad \frac{(u - 1)^2}{u} \leq 0 \quad \Longleftrightarrow \quad u < 0 \quad \text{or} \quad u = 1$$

If $u = 1$, then $a = b$: $x^2 - x + 20 = x^2 - 8x + 17$, so $x = -\frac{3}{7}$.

If $u < 0$, then a and b are on different sides of 1. Now $a < 1$ if and only if $(x - 6)(x + 5) < 0$, that is, $-5 < x < 6$, and $b < 1$ if and only if $-3 < x < 11$.

$a < 1 < b$: $-5 < x < 6$ and $(x < -3 \text{ or } x > 11)$, so $-5 < x < -3$.

$b < 1 < a$: $-3 < x < 11$ and $(x < -5 \text{ or } x > 6)$, so $6 < x < 11$.

So the solution is $-5 < x < -3$, $6 < x < 11$, or $x = -\frac{3}{7}$.