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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Three numbers are consecutive terms of an arithmetic sequence, and the middle one is 2. Find the other two numbers if the squares of the three numbers are consecutive terms of a geometric sequence.

Solution: Let the numbers be  $2 - d$ ,  $2$ ,  $2 + d$ . Their squares form a geometric sequence if

$$\begin{aligned} 2^4 &= (2 - d)^2 (2 + d)^2 \\ 16 &= (4 - d^2)^2 \\ 4 - d^2 &= \pm 4 \end{aligned}$$

So  $d = 0$  or  $d^2 = 8$ . If  $d = 0$ , all three numbers are 2. If  $d = \pm 2\sqrt{2}$ , the other two numbers are  $2 - 2\sqrt{2}$  and  $2 + 2\sqrt{2}$ . (Check: their squares are  $4(3 - 2\sqrt{2})$ ,  $4$ ,  $4(3 + 2\sqrt{2})$ , and  $4(3 - 2\sqrt{2}) \cdot 4(3 + 2\sqrt{2}) = 16 \cdot 1 = 4^2$ .)

2. a) Find the exact value of  $\tan 15^\circ$ .

b) The legs  $a$  and  $b$  of a right triangle satisfy  $(a - b)^2 = 2ab$ . Find the exact values of the acute angles of the triangle.

Solution: a) By the subtraction formula,

$$\tan 15^\circ = \tan (45^\circ - 30^\circ) = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

b) Expanding,  $a^2 - 2ab + b^2 = 2ab$ , so  $a^2 + b^2 = 4ab$ . Dividing by  $b^2$  and letting  $t = \frac{a}{b}$ :

$$t^2 - 4t + 1 = 0 \quad \text{so} \quad t = 2 \pm \sqrt{3}$$

Here  $t$  is the tangent of the angle opposite  $a$ . By part a),  $\tan 15^\circ = 2 - \sqrt{3}$ , and  $\tan 75^\circ = \frac{1}{\tan 15^\circ} = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$ .

So the acute angles are  $15^\circ$  and  $75^\circ$ .

3. Consider the equation  $x^2 + mx - 2m^2 + 3m - 1 = 0$ , where  $m$  is a real parameter.

- Find the values of  $m$  for which one root of the equation is twice the other root.
- Find the value of  $m$  for which the product of the two roots is as large as possible.

Solution: The discriminant is

$$m^2 - 4(-2m^2 + 3m - 1) = 9m^2 - 12m + 4 = (3m - 2)^2 \geq 0$$

so the roots are always real, and

$$x = \frac{-m \pm (3m - 2)}{2}, \quad \text{that is,} \quad x_1 = m - 1 \quad \text{and} \quad x_2 = 1 - 2m$$

a) If  $x_1 = 2x_2$ :  $m - 1 = 2 - 4m$ , so  $m = \frac{3}{5}$  (the roots are  $-\frac{2}{5}$  and  $-\frac{1}{5}$ ). If  $x_2 = 2x_1$ :  $1 - 2m = 2m - 2$ , so  $m = \frac{3}{4}$  (the roots are  $-\frac{1}{4}$  and  $-\frac{1}{2}$ ).

b) The product of the roots is  $-2m^2 + 3m - 1$ . This is a quadratic function of  $m$  opening downward, with maximum at  $m = -\frac{3}{2 \cdot (-2)} = \frac{3}{4}$ . The maximal product is  $-2 \cdot \frac{9}{16} + \frac{9}{4} - 1 = \frac{1}{8}$ .

4. Find the equation of the circle that is tangent to the line  $x + y = 0$  at the origin, and is also tangent to the line  $x = 1$ .

Solution: The center is on the line through the origin perpendicular to  $x + y = 0$ , which is  $y = x$ . So the center is  $(t, t)$ , and the radius is its distance from the origin,  $r = \sqrt{2}|t|$ . The circle is tangent to  $x = 1$ , so its distance from the center is also  $r$ :

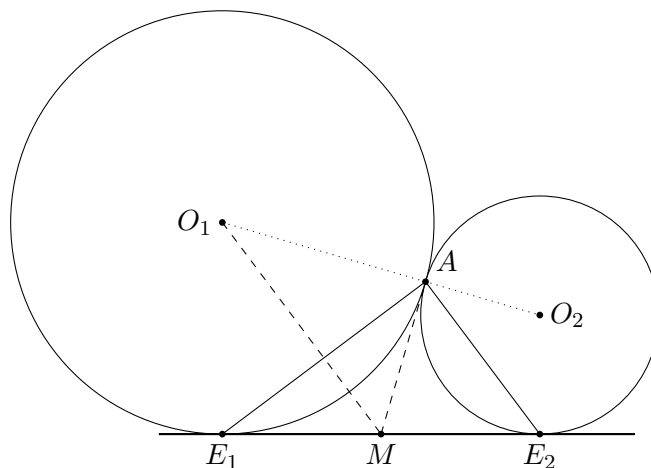
$$\begin{aligned} |t - 1| &= \sqrt{2}|t| && \text{square both sides} \\ t^2 - 2t + 1 &= 2t^2 \\ t^2 + 2t - 1 &= 0 \\ t &= -1 \pm \sqrt{2} \end{aligned}$$

If  $t = \sqrt{2} - 1$ , then  $r = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$ ; if  $t = -1 - \sqrt{2}$ , then  $r = 2 + \sqrt{2}$ . So there are two such circles:

$$\begin{aligned} (x - \sqrt{2} + 1)^2 + (y - \sqrt{2} + 1)^2 &= (2 - \sqrt{2})^2 = 6 - 4\sqrt{2} \\ (x + \sqrt{2} + 1)^2 + (y + \sqrt{2} + 1)^2 &= (2 + \sqrt{2})^2 = 6 + 4\sqrt{2} \end{aligned}$$

5. The circles  $k_1$  and  $k_2$  are tangent to each other from outside at point  $A$ . A common tangent line of the two circles touches  $k_1$  at  $E_1$  and  $k_2$  at  $E_2$ . Find the radii of the circles if  $AE_1 = 8$  and  $AE_2 = 6$  units.

Solution: Let  $O_1$  and  $O_2$  be the centers and  $r_1, r_2$  the radii. The common tangent of the circles at  $A$  intersects  $E_1E_2$  at a point  $M$ . The two tangent segments from  $M$  to each circle are equal, so  $ME_1 = MA = ME_2$ .



So  $M$  is the midpoint of  $E_1E_2$ , and  $A$  is on the circle with diameter  $E_1E_2$ . By Thales' theorem,  $\angle E_1AE_2 = 90^\circ$ , so  $E_1E_2 = \sqrt{8^2 + 6^2} = 10$  and  $ME_1 = MA = 5$ .

The quadrilateral  $MAO_1E_1$  is a kite, symmetric about  $MO_1$ , with right angles at  $A$  and  $E_1$ . Its diagonal  $AE_1$  is perpendicular to  $MO_1$ , and half of  $AE_1$  is the height of the right triangle  $ME_1O_1$  belonging to the hypotenuse. Computing the area of this triangle in two ways,

$$\begin{aligned} \frac{AE_1}{2} \cdot MO_1 &= ME_1 \cdot O_1E_1 \\ 4\sqrt{25 + r_1^2} &= 5r_1 && \text{square both sides} \\ 400 + 16r_1^2 &= 25r_1^2 \\ r_1 &= \frac{20}{3} \end{aligned}$$

The same way,  $3\sqrt{25 + r_2^2} = 5r_2$ , so  $225 = 16r_2^2$  and  $r_2 = \frac{15}{4}$ . The radii are  $\frac{20}{3} \approx 6.67$  and  $\frac{15}{4} = 3.75$  units. (Check:  $E_1E_2 = 2\sqrt{r_1r_2} = 2\sqrt{25} = 10$ .)

6. For which real numbers  $x$  is the value of the expression

$$\frac{3x^2 - 7x + 8}{x^2 + 1}$$

greater than 1 and at most 2?

Solution: The denominator  $x^2 + 1$  is positive, so we can multiply by it without changing the direction of the inequalities.

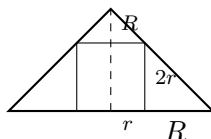
The condition "greater than 1":  $3x^2 - 7x + 8 > x^2 + 1$ , that is,  $2x^2 - 7x + 7 > 0$ . The discriminant is  $49 - 56 < 0$ , so this is true for every  $x$ .

The condition "at most 2":  $3x^2 - 7x + 8 \leq 2x^2 + 2$ , that is,  $x^2 - 7x + 6 = (x - 1)(x - 6) \leq 0$ , so  $1 \leq x \leq 6$ .

So the answer is  $1 \leq x \leq 6$ .

7. The radius of the base of a right circular cone is equal to its height. We inscribe a right circular cylinder in the cone so that the height of the cylinder is equal to the diameter of its base. What percent of the surface area of the cone is the surface area of the cylinder? What percent of the volume of the cone is the volume of the cylinder?

Solution: Let  $R$  be the radius and the height of the cone, and let  $r$  be the radius of the cylinder, so its height is  $2r$ . Consider a cross-section through the axis.



At height  $y$ , the radius of the cone is  $R - y$  (since the radius equals the height). The top circle of the cylinder is at height  $2r$  and is on the cone, so  $r = R - 2r$ , and  $r = \frac{R}{3}$ .

The volumes:

$$V_{\text{cyl}} = \pi r^2 \cdot 2r = \frac{2\pi R^3}{27}, \quad V_{\text{cone}} = \frac{\pi R^2 \cdot R}{3} = \frac{\pi R^3}{3}, \quad \frac{V_{\text{cyl}}}{V_{\text{cone}}} = \frac{2}{9} \approx 22.2\%$$

The surface areas (the slant height of the cone is  $R\sqrt{2}$ ):

$$A_{\text{cyl}} = 2\pi r^2 + 2\pi r \cdot 2r = 6\pi r^2 = \frac{2\pi R^2}{3}, \quad A_{\text{cone}} = \pi R^2 + \pi R \cdot R\sqrt{2} = \pi R^2 (1 + \sqrt{2})$$

$$\frac{A_{\text{cyl}}}{A_{\text{cone}}} = \frac{2}{3(1 + \sqrt{2})} = \frac{2(\sqrt{2} - 1)}{3} \approx 27.6\%$$

8. Find the values of the parameter  $m$  for which the equation

$$\tan 2x = m \cot x$$

has a solution. Solve the equation if  $m = 1$ .

Solution: We need  $\sin x \neq 0$  (for  $\cot x$ ) and  $\cos 2x \neq 0$  (for  $\tan 2x$ ).

First, notice that if  $x = \frac{\pi}{2} + k\pi$ , then  $\tan 2x = \tan(\pi + 2k\pi) = 0$  and  $\cot x = 0$ , so both sides are 0, and these values are solutions for every  $m$ .

Now suppose  $\cos x \neq 0$ , and let  $t = \tan x$  ( $t \neq 0$ ). Then  $\tan 2x = \frac{2t}{1 - t^2}$  and  $\cot x = \frac{1}{t}$ , so

$$\begin{aligned} \frac{2t}{1 - t^2} &= \frac{m}{t} \\ 2t^2 &= m(1 - t^2) \\ (m + 2)t^2 &= m \end{aligned}$$

If  $m = -2$ , there is no solution of this kind. Otherwise  $t^2 = \frac{m}{m + 2}$ , and we need  $t^2 > 0$  (and  $t^2 \neq 1$ , which is automatic). So there are solutions with  $\tan x$  defined if and only if  $\frac{m}{m + 2} > 0$ , that is,  $m > 0$  or  $m < -2$ .

In summary:  $x = \frac{\pi}{2} + k\pi$  is a solution for every  $m$ ; if we look only for solutions where  $\tan x$  is defined (which is probably what the problem intended), then there are solutions if and only if  $m > 0$  or  $m < -2$ .

If  $m = 1$ :  $t^2 = \frac{1}{3}$ , so  $\tan x = \pm \frac{1}{\sqrt{3}}$  and  $x = \pm \frac{\pi}{6} + k\pi$ . Together with  $x = \frac{\pi}{2} + k\pi$ , these are all the solutions.

(Check:  $\tan \frac{\pi}{3} = \sqrt{3} = \cot \frac{\pi}{6}$ .)