

author: Imre Rabai

translated from: KoMaL, November 1985, pages 357-358

Pdf file: KoMaL, November 1985, pages 357-358

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve each of the following equations.

$$a) 2^{2x^2+6x-8} = 4^{\frac{x-1}{x+4}}$$

$$b) \sqrt{x^2 + 3x - 4} = \left(\frac{x-1}{x+4}\right)^{\frac{1}{2}}$$

$$c) \log(x^2 + 3x - 4) = \log\left(\frac{x-1}{x+4}\right)$$

Solution: a) domain: $x \neq -4$

$$\begin{aligned} 2^{2x^2+6x-8} &= 4^{\frac{x-1}{x+4}} \\ \left(4^{\frac{1}{2}}\right)^{2(x^2+3x-4)} &= 4^{\frac{x-1}{x+4}} \\ 4^{x^2+3x-4} &= 4^{\frac{x-1}{x+4}} \end{aligned}$$

Since the function $f(x) = 4^x$ is one-to-one,

$$\begin{aligned} x^2 + 3x - 4 &= \frac{x-1}{x+4} \\ (x+4)(x-1) &= \frac{x-1}{x+4} \\ (x+4)(x-1) - \frac{x-1}{x+4} &= 0 \\ (x-1)\left(x+4 - \frac{1}{x+4}\right) &= 0 \quad \text{multiply by } x+4 \\ (x-1)\left((x+4)^2 - 1\right) &= 0 \\ (x-1)(x+4+1)(x+4-1) &= 0 \\ (x-1)(x+5)(x+3) &= 0 \\ x_1 = 1 \quad x_2 = -5 \quad x = -3 \end{aligned}$$

We should check: If $x = 1$

$$\begin{aligned} \text{LHS} &= 2^{2x^2+6x-8} = 2^{2+6-8} = 2^0 = 1 \\ \text{RHS} &= 4^{\frac{x-1}{x+4}} = 4^{\frac{1-1}{1+4}} = 4^0 = 1 \end{aligned}$$

If $x = -5$

$$\begin{aligned}\text{LHS} &= 2^{2x^2+6x-8} = 2^{2 \cdot 25 + 6(-5) - 8} = 2^{12} \\ \text{RHS} &= 4^{\frac{x-1}{x+4}} = 4^{\frac{-5-1}{-5+4}} = 4^6 = 2^{12}\end{aligned}$$

If $x = -3$

$$\begin{aligned}\text{LHS} &= 2^{2x^2+6x-8} = 2^{2 \cdot 9 + 6(-3) - 8} = 2^{-8} = \frac{1}{2^8} \\ \text{RHS} &= 4^{\frac{x-1}{x+4}} = 4^{\frac{-3-1}{-3+4}} = 4^{-4} = \frac{1}{2^8}\end{aligned}$$

So all three work.

$$\text{b) } \sqrt{x^2 + 3x - 4} = \left(\frac{x-1}{x+4} \right)^{\frac{1}{2}}$$

Solution: domain:

$$\begin{aligned}x^2 + 3x - 4 &\geq 0 \\ (x+4)(x-1) &\geq 0 \\ x &\leq -4 \text{ or } x \geq 1 \\ \frac{x-1}{x+4} &\geq 0 \\ x < -4 \text{ or } x \geq 1 &\text{ this is the domain}\end{aligned}$$

$$\begin{aligned}\sqrt{x^2 + 3x - 4} &= \left(\frac{x-1}{x+4} \right)^{\frac{1}{2}} && \text{square} \\ x^2 + 3x - 4 &= \frac{x-1}{x+4} \\ (x+4)(x-1) &= \frac{x-1}{x+4} \\ (x+4)(x-1) - \frac{x-1}{x+4} &= 0 \\ (x-1) \left(x+4 - \frac{1}{x+4} \right) &= 0 && x \neq -4 \\ (x-1) \left((x+4)^2 - 1 \right) &= 0 \\ (x-1)(x+4+1)(x+4-1) &= 0 \\ (x-1)(x+5)(x+3) &= 0 \\ x_1 = 1 \quad x_2 = -5 \quad x_3 = -3\end{aligned}$$

Only -5 and 1 are in the domain. We check: if $x = -5$

$$\begin{aligned}\text{LHS} &= \sqrt{x^2 + 3x - 4} = \sqrt{25 + 3(-5) - 4} = \sqrt{6} \\ \text{RHS} &= \left(\frac{x-1}{x+4} \right)^{\frac{1}{2}} = \left(\frac{-5-1}{-5+4} \right)^{\frac{1}{2}} = 6^{\frac{1}{2}} = \sqrt{6}\end{aligned}$$

If $x = 1$, then

$$\begin{aligned}\text{LHS} &= \sqrt{x^2 + 3x - 4} = \sqrt{1 + 3 \cdot 1 - 4} = \sqrt{0} = 0 \\ \text{RHS} &= \left(\frac{x-1}{x+4} \right)^{\frac{1}{2}} = \left(\frac{1-1}{1+4} \right)^{\frac{1}{2}} = 0^{\frac{1}{2}} = 0\end{aligned}$$

c) $\log(x^2 + 3x - 4) = \log\left(\frac{x-1}{x+4}\right)$

Solution: the expressions are the same. Domain:

$$x^2 + 3x - 4 > 0 \implies x < -4 \text{ or } x > 1$$

$$\frac{x-1}{x+4} > 0 \implies x < -4 \text{ or } x > 1$$

$$\log(x^2 + 3x - 4) = \log\left(\frac{x-1}{x+4}\right) \quad f(x) = \log x \text{ is one-to-one}$$

$$x^2 + 3x - 4 = \frac{x-1}{x+4}$$

$$(x+4)(x-1) - \frac{x-1}{x+4} = 0$$

$$(x-1)\left(x+4 - \frac{1}{x+4}\right) = 0 \quad x \neq -4$$

$$(x-1)((x+4)^2 - 1) = 0$$

$$(x-1)(x+4+1)(x+4-1) = 0$$

$$(x-1)(x+5)(x+3) = 0$$

$$x_1 = 1 \quad x_2 = -5 \quad x_3 = -3$$

This time only -5 is in the domain.

2. Compute the exact value of each of the following expressions.

a) $\tan 110^\circ + \cot 20^\circ$

b) $\frac{\sqrt[3]{9^2} \left(\frac{1}{3}\right)^6 \cdot 3 \left(\sqrt[3]{3}\right)^4}{\left(\sqrt[3]{3}\right)^{-1} \cdot 27^{-\frac{2}{3}}}$

c) $27^{\frac{1}{\log_5 3}} - 10^{2-\log 4}$

a) $\tan 110^\circ + \cot 20^\circ$

Solution:

$$\begin{aligned} \tan 110^\circ + \cot 20^\circ &= \tan 110^\circ + \frac{1}{\tan 20^\circ} = \tan(110^\circ - 180^\circ) + \frac{1}{\tan 20^\circ} = \tan(-70^\circ) + \frac{1}{\tan 20^\circ} \\ &= -\tan 70^\circ + \tan 70^\circ = 0 \end{aligned}$$

b) $\frac{\sqrt[3]{9^2} \left(\frac{1}{3}\right)^6 \cdot 3 \left(\sqrt[3]{3}\right)^4}{\left(\sqrt[3]{3}\right)^{-1} \cdot 27^{-\frac{2}{3}}}$

Solution:

$$\frac{\sqrt[3]{9^2} \left(\frac{1}{3}\right)^6 \cdot 3 \left(\sqrt[3]{3}\right)^4}{\left(\sqrt[3]{3}\right)^{-1} \cdot 27^{-\frac{2}{3}}} = \frac{(3^2)^{\frac{2}{3}} 3^{-6} \cdot 3 \cdot 3^{\frac{4}{3}}}{3^{\frac{-1}{3}} \cdot (3^3)^{-\frac{2}{3}}} = \frac{3^{\frac{4}{3}-6+1+\frac{4}{3}}}{3^{-\frac{1}{3}+(-2)}} = \frac{3^{\frac{8}{3}-5}}{3^{\frac{-7}{3}}} = \frac{3^{\frac{8-15}{3}}}{3^{\frac{-7}{3}}} = \frac{3^{\frac{-7}{3}}}{3^{\frac{-7}{3}}} = 1$$

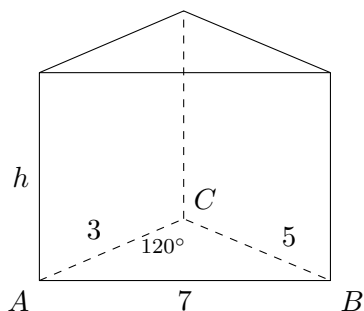
c) $27^{\frac{1}{\log_5 3}} - 10^{2-\log 4}$

Solution:

$$27^{\frac{1}{\log_5 3}} - 10^{2-\log 4} = 27^{\log_3 5} - \frac{10^2}{10^{\log 4}} = (3^3)^{\log_3 5} - \frac{100}{4} = \left(3^{\log_3 5}\right)^3 - 25 = 5^3 - 25 = 125 - 25 = 100$$

3. The base of a straight prism is triangle ABC , in which $AC = 3$ unit and $BC = 5$ unit, and the angle at vertex C is 120° . The face with the greatest area is 35 unit^2 . Compute the volume and surface area of the prism.

Solution:



The greatest side of the base is AB . We can find the longest side via the Law of Cosines.

$$\begin{aligned} 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos 120^\circ &= AB^2 \\ 9 + 25 - 30 \left(-\frac{1}{2} \right) &= AB^2 \\ 34 + 15 &= AB^2 \\ 49 &= AB^2 \\ \pm 7 &= AB \implies AB = 7 \end{aligned}$$

The area of triangle ABC is

$$A = \frac{1}{2} (AC) (BC) \sin 120^\circ = \frac{1}{2} \cdot 3 \cdot 5 \cdot \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{4} \text{ unit}^2$$

The largest face is 7 units by the height h of the prism, so $7h = 35$. Thus the height is 5 units. So the surface area is

$$S = 2A + 5 \cdot 3 + 5 \cdot 5 + 5 \cdot 7 = 2 \cdot \frac{15\sqrt{3}}{4} + 15 + 25 + 35 = \frac{15\sqrt{3}}{2} + 75 \text{ unit}^2 \approx 87.9904 \text{ unit}^2$$

The volume:

$$V = Ah = \left(\frac{15\sqrt{3}}{4} \text{ unit}^2 \right) (5 \text{ unit}) = \frac{75\sqrt{3}}{4} \text{ unit}^3 \approx 32.475953 \text{ unit}^3$$

4. The vertices of triangle ABC are given as $A(2, 0)$, $B(-2, 0)$, and $C(1, 3)$. The circle $x^2 + y^2 = 4$ intersects side AC in point R , and intersects the side BC in point S . Find the equation for the circle that contains points C , R , and S .

Solution: To find R . First, side AC is

$$m = \frac{3}{-1} = -3 \quad -3(x - 2) = y \quad y = -3x + 6$$

$$\begin{aligned} x^2 + y^2 &= 4 \\ y &= -3x + 6 \end{aligned}$$

$$\begin{aligned} x^2 + (-3x + 6)^2 &= 4 \\ x^2 + 9x^2 - 36x + 36 - 4 &= 0 \\ 10x^2 - 36x + 32 &= 0 \\ 5x^2 - 18x + 16 &= 0 \end{aligned}$$

$$x_{1,2} = \frac{18 \pm \sqrt{18^2 - 4 \cdot 5 \cdot 16}}{10} = \frac{18 \pm \sqrt{324 - 320}}{10} = \frac{18 \pm \sqrt{4}}{10} = \frac{18 \pm 2}{10} = \begin{cases} \frac{20}{10} = 2 \implies y = 0 \\ \frac{16}{10} = 1.6 \implies y = -4.8 + 6 = 1.2 \end{cases}$$

So R is $R(1.6, 1.2)$

Now to find side BC .

$$m = \frac{3}{3} = 1 \quad x + 2 = y \quad y = x + 2$$

To find S :

$$\begin{aligned} x^2 + y^2 &= 4 \\ y &= x + 2 \end{aligned}$$

$$\begin{aligned} x^2 + (x + 2)^2 &= 4 \\ x^2 + x^2 + 4x + 4 &= 4 \\ 2x^2 + 4x &= 0 \\ 2x(x + 2) &= 0 \end{aligned}$$

$$x_1 = 0 \implies y_1 = 2 \quad x_2 = -2 \implies y = 0$$

So S is $S(0, 2)$. We now need the circle that contains $C(1, 3)$ and $R(1.6, 1.2)$ and $S(0, 2)$.

It would be very nice if this triangle was right. Let's see.

$$\begin{aligned} m_{CR} &= \frac{3 - 1.2}{1 - 1.6} = \frac{1.8}{-0.6} = -3 \\ m_{RS} &= \frac{1.2 - 2}{1.6 - 0} = \frac{-0.8}{1.6} = -\frac{1}{2} \\ m_{CS} &= \frac{3 - 2}{1 - 0} = \frac{1}{1} = 1 \end{aligned}$$

It is not. The center of the circumscribed circle is the intersection of perpendicular bisectors.

$$\begin{aligned} M_{CR} &= \left(\frac{1 + 1.6}{2}, \frac{3 + 1.2}{2} \right) = (1.3, 2.1) \text{ and } m_p = \frac{1}{3} \\ \frac{1}{3}(x - 1.3) &= y - 2.1 \\ x - 1.3 &= 3y - 6.3 \\ x &= 3y - 5 \end{aligned}$$

$$\begin{aligned} M_{CS} &= \left(\frac{1 + 0}{2}, \frac{3 + 2}{2} \right) = \left(\frac{1}{2}, \frac{5}{2} \right) \text{ and } m_p = -1 \\ -1 \left(x - \frac{1}{2} \right) &= y - \frac{5}{2} \\ -x + \frac{1}{2} &= y - \frac{5}{2} \\ -x + 3 &= y \end{aligned}$$

Intersect these lines for the center:

$$\begin{aligned} y &= -x + 3 \\ x &= 3y - 5 \end{aligned}$$

$$\begin{aligned}y &= -(3y - 5) + 3 \\y &= -3y + 8 \\4y &= 8 \\y &= 2 \implies x = 3 \cdot 2 - 5 = 1\end{aligned}$$

The radius is the distance between the center $P(1, 2)$ and any of the points (we'll take $C(1, 3)$).

$$r = \sqrt{(1-1)^2 + (2-3)^2} = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$$

and so the equation of the circle is

$$(x-1)^2 + (y-2)^2 = 1$$

We check. Is $C(1, 3)$ on the circle?

$$(1-1)^2 + (3-2)^2 = 1^2 = 1 \text{ works}$$

Is $R(1.6, 1.2)$ on the circle?

$$(1.6-1)^2 + (1.2-2)^2 = 0.6^2 + (-0.8)^2 = 1 \text{ works}$$

and $S(0, 2)$

$$(0-1)^2 + (2-2)^2 = 1 \text{ works}$$

5. Find the quadratic equation in which the quadratic coefficient is 1, one of the solutions is 2, and the other solution is equal to the discriminant of the equation.

Solution: Suppose the equation is $x^2 + bx + c = 0$.

One of the solutions is 2:

$$\begin{aligned}2^2 + b \cdot 2 + c &= 0 \\2b + c &= -4 \implies c = -2b - 4\end{aligned}$$

The other solution is the same as the discriminant. The discriminant is $D = b^2 - 4ac = b^2 - 4c$.

Recall that the sum of the two solutions is $-\frac{b}{a}$ and their product is $\frac{c}{a}$.

$$\begin{aligned}D &= b^2 - 4c \\2 + D &= -b & b &= -D - 2 \\2D &= c & c &= 2D\end{aligned}$$

$$\begin{aligned}D &= b^2 - 4c = (-D - 2)^2 - 4(2D) \\D &= (-D - 2)^2 - 8D \\D &= D^2 + 4D + 4 - 8D \\0 &= D^2 - 5D + 4 \\0 &= (D - 4)(D - 1)\end{aligned}$$

$$D_1 = 4 \quad D_2 = 1$$

If $D = 4$, then the equation is

$$\begin{aligned}(x-2)(x-4) &= 0 \\x^2 - 6x + 8 &= 0\end{aligned}$$

The discriminant is $36 - 32 = 4$. Works. If $D = 1$, then the equation is

$$\begin{aligned}(x-2)(x-1) &= 0 \\ x^2 - 3x + 2 &= 0\end{aligned}$$

The discriminant is $9 - 8 = 1$. Works.

6. Prove that if $\sin(\alpha + \beta) = 0$, then $\cos(\alpha + 2\beta) = \cos \alpha$. Is the converse of the statement true?

Solution: We have:

$$\begin{aligned}\sin(\alpha + \beta) &= 0 \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta &= 0\end{aligned}$$

Need:

$$\begin{aligned}\cos(\alpha + 2\beta) - \cos \alpha &= 0 \\ \cos \alpha \cos 2\beta - \sin \alpha \sin 2\beta - \cos \alpha &= 0 \\ \cos \alpha (\cos^2 \beta - \sin^2 \beta) - \sin \alpha (2 \sin \beta \cos \beta) - \cos \alpha &= 0 \\ \cos \alpha (\cos^2 \beta - \sin^2 \beta - 1) - \sin \alpha (2 \sin \beta \cos \beta) &= 0 \\ \cos \alpha ((\cos^2 \beta - 1) - \sin^2 \beta) - \sin \alpha (2 \sin \beta \cos \beta) &= 0 \\ \cos \alpha (-2 \sin^2 \beta) - \sin \alpha (2 \sin \beta \cos \beta) &= 0 \\ -2 \cos \alpha \sin^2 \beta - 2 \sin \alpha \sin \beta \cos \beta &= 0 \\ \cos \alpha \sin^2 \beta + \sin \alpha \sin \beta \cos \beta &= 0 \\ \sin \beta (\cos \alpha \sin \beta + \sin \alpha \cos \beta) &= 0 \\ \sin \beta (\sin(\alpha + \beta)) &= 0\end{aligned}$$

$$\sin \beta = 0 \text{ or } \sin(\alpha + \beta) = 0$$

So, $\cos(\alpha + 2\beta) = \cos \alpha$ is equivalent to $\sin \beta = 0$ or $\sin(\alpha + \beta) = 0$. The converse is not true. (Try $\beta = 0$ and $\alpha = \frac{\pi}{2}$)

Other way:

$$\begin{aligned}\sin(\alpha + \beta) &= 0 \\ \alpha + \beta &= k\pi \text{ where } k \in \mathbb{Z} \\ \beta &= -\alpha + k\pi \text{ where } k \in \mathbb{Z} \\ 2\beta &= -2\alpha + 2k\pi \text{ where } k \in \mathbb{Z} \\ \alpha + 2\beta &= -\alpha + 2k\pi \text{ where } k \in \mathbb{Z} \\ \cos(\alpha + 2\beta) &= \cos(-\alpha + 2k\pi) \text{ where } k \in \mathbb{Z} \\ \cos(\alpha + 2\beta) &= \cos(-\alpha) \\ \cos(\alpha + 2\beta) &= \cos \alpha\end{aligned}$$

Notice that the step from $2\beta = -2\alpha + 2k\pi$ to $\alpha + 2\beta = -\alpha + 2k\pi$ only works in one direction; this is why the converse is not true.

7. The height belonging to the base in an isosceles triangle is 8 units long. The triangle's inscribed circle has a radius of 2 units long. Find the radius of the circle that is tangent to the legs and the inscribed circle.

Solution: Let x denote the radius of the circle.



$$(2^x)_{1,2} = \frac{2a \pm \sqrt{(2a)^2 - 4a(a-4)}}{2(a-4)} = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 16a}}{2(a-4)} = \frac{2a \pm \sqrt{16a}}{2(a-4)} = \frac{a \pm \sqrt{4a}}{a-4}$$

First the discriminant: If $a < 0$, there is no solution. Suppose now that $a \geq 0$. Then there are solutions for 2^x but that doesn't necessarily mean solutions for x . Because $2^x > 0$, a solution only exists if 2^x is positive. We need to solve:

$$\frac{a \pm \sqrt{4a}}{a - 4} > 0$$

Case 1. If $a > 4$. Then we need

$$a + \sqrt{4a} > 0 \quad \text{and} \quad a - \sqrt{4a} > 0$$

The first inequality is automatically true. The second one:

$$\begin{aligned} a - \sqrt{4a} &> 0 \\ \sqrt{a}\sqrt{a} - \sqrt{4a} &> 0 \\ \sqrt{a}(\sqrt{a} - \sqrt{4}) &> 0 \\ \sqrt{a}(\sqrt{a} - 2) &> 0 \quad \sqrt{a} > 0 \\ \sqrt{a} - 2 &> 0 \\ \sqrt{a} &> 2 \\ a &> 4 \end{aligned}$$

So $a > 4$ means that we have two solutions for x :

$$x_1 = \log_2 \left(\frac{a + \sqrt{4a}}{a - 4} \right) \quad \text{and} \quad x_2 = \log_2 \left(\frac{a - \sqrt{4a}}{a - 4} \right)$$

Case 2. If $0 < a < 4$. Then $\frac{a + \sqrt{4a}}{a - 4}$ is negative (positive numerator and negative denominator)

For the other solution, we need that the numerator is also negative:

$$\begin{aligned} a - \sqrt{4a} &< 0 \\ a &< \sqrt{4a} \\ \sqrt{a}\sqrt{a} &< \sqrt{4}\sqrt{a} \quad \text{divide by } \sqrt{a} > 0 \\ \sqrt{a} &< 2 \\ a &< 4 \end{aligned}$$

and so if $0 < a < 4$, then there is just one solution:

$$x = \log_2 \left(\frac{a - \sqrt{4a}}{a - 4} \right)$$

Finally, what if $a = 0$?

$$\begin{aligned} -4 \cdot 2^x + \frac{0}{2^x} &= 0 \\ -4 \cdot 2^x &= 0 \\ 2^x &= 0 \end{aligned}$$

There is no solution, since $2^x > 0$.

And if $a = 4$, then the equation is not quadratic:

$$\begin{aligned} 0 \cdot 2^x + 4 \cdot 2^{-x} &= 8 \\ 2^{-x} &= 2 \\ x &= -1 \end{aligned}$$

To summarize: if $a \leq 0$, there is no solution; if $0 < a < 4$, there is one solution, $x = \log_2 \left(\frac{a - \sqrt{4a}}{a - 4} \right)$; if $a = 4$, there is one solution, $x = -1$; and if $a > 4$, there are two solutions, $x_{1,2} = \log_2 \left(\frac{a \pm \sqrt{4a}}{a - 4} \right)$.