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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find all positive solutions of the equation  $\sqrt{x+2} = -x$ .

Solution: Notice the domain of the left-hand side,  $x \geq -2$  and the range of the right-hand side. Since the left-hand side is a square root, it must be non-negative. So  $-x \geq 0$  implies  $x \leq 0$ . So, any solution must be in the interval  $[-2, 0]$ .

Square both sides:

$$\begin{aligned}\sqrt{x+2} &= -x && \text{square both sides} \\ x+2 &= x^2 \\ 0 &= x^2 - x - 2 \\ 0 &= (x-2)(x+1) \\ x_1 &= 2 \text{ and } x_2 = -1\end{aligned}$$

We check and find that 2 does not work (it's also not in our domain).  $-1$  works and is the only solution.

2. Two angles in a triangle have sines 0.7431 and 0.6691. Compute the cosine of the third angle.

Solution 2: We introduce notation:  $\sin \alpha = 0.7431$  and  $\sin \beta = 0.6691$ .

$$\begin{aligned}\cos \alpha &= \pm \sqrt{1 - \sin^2 \alpha} = \pm \sqrt{1 - (0.7431)^2} = \pm \sqrt{1 - (0.7431)^2} = \pm \sqrt{0.44780239} \\ &\approx \pm 0.66918 \\ \cos \beta &= \pm \sqrt{1 - \sin^2 \beta} = \pm \sqrt{1 - (0.6691)^2} = \pm \sqrt{0.55230519} \approx \pm 0.743172\end{aligned}$$

Considering the order between  $\sin \alpha$  and  $\sin \beta$ , we conclude that  $\alpha > \beta$  and so only  $\alpha$  could be the greatest angle in the triangle. Thus,  $\beta$  must be an acute angle and so its cosine must be positive. Thus, there are two cases.

Case 1.  $\alpha$  is obtuse:  $\sin \alpha = 0.7431$ ,  $\cos \alpha = -0.6692$ ,  $\sin \beta = 0.6691$ ,  $\cos \beta = 0.7431$

Case 2.  $\alpha$  is acute:  $\sin \alpha = 0.7431$ ,  $\cos \alpha = 0.6692$ ,  $\sin \beta = 0.6691$ ,  $\cos \beta = 0.7431$

In both cases, the computation will be as follows:

$$\cos \gamma = \cos(180^\circ - (\alpha + \beta)) = -\cos(\alpha + \beta) = -(\cos \alpha \cos \beta - \sin \alpha \sin \beta) = -\cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Case 1:

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta = -(-0.6692)(0.7431) + (0.7431)(0.6691) \approx 0.9945$$

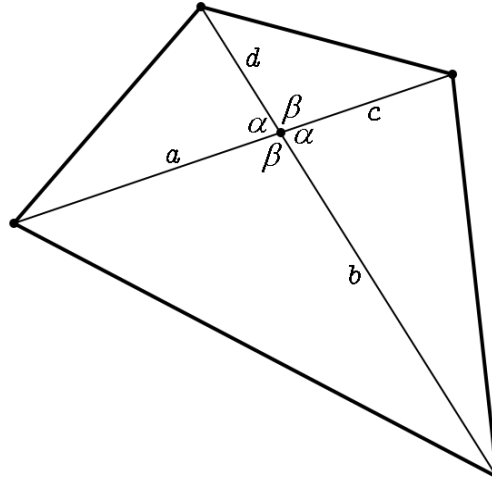
Case 2:

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta = -(0.6692)(0.7431) + (0.7431)(0.6691) \approx 0$$

This second result is not such a surprise. Ignoring the very slight inconsistencies, basically  $\cos \alpha = \sin \beta$ . That means that  $\alpha$  and  $\beta$  are complementary, and so  $\gamma = 90^\circ$ .

3. The diagonals of a complex quadrilateral split it into four triangles. The areas of three such triangles are  $1 \text{ cm}^2$ ,  $2 \text{ cm}^2$ , and  $3 \text{ cm}^2$ . Find the area of the fourth triangle.

Solution: Recall that the area of a triangle is  $A = \frac{1}{2}ab \sin \gamma$  where  $\gamma$  is the angle between sides  $a$  and  $b$ . Let  $a, b, c, d$  be the line segments into which the diagonals cut each other.



Also notice that  $\beta = 180^\circ - \alpha$  and so  $\sin \beta = \sin \alpha$ .

$$\begin{aligned} \frac{1}{2}cd \sin \alpha &= 1 \\ \frac{1}{2}ad \sin \alpha &= 2 \\ \frac{1}{2}ab \sin \alpha &= 3 \\ \frac{1}{2}bc \sin \alpha &= ? \end{aligned}$$

$$\begin{aligned} cd &= \frac{2}{\sin \alpha} \\ ad &= \frac{4}{\sin \alpha} \\ ab &= \frac{6}{\sin \alpha} \end{aligned}$$

$$\frac{1}{2}bc \sin \alpha = \frac{1}{2} \sin \alpha (bc) = \frac{1}{2} \sin \alpha \frac{(ab)(cd)}{ad} = \frac{1}{2} \sin \alpha \frac{\left(\frac{6}{\sin \alpha}\right) \left(\frac{2}{\sin \alpha}\right)}{\frac{4}{\sin \alpha}} = \frac{1}{2} \cdot \frac{6 \left(\frac{2}{\sin \alpha}\right)}{\frac{4}{\sin \alpha}} = \frac{1}{2} \cdot \frac{6 \cdot 2}{4} = \frac{3}{2}$$

4. Solve the given equation.

$$\sqrt{\sin x \cos x} \cdot (1 - \log(16 - x^2)) = 0$$

Solution: Consider the domain of this equation first.

$$\sin x \cos x \geq 0 \implies \sin x, \cos x \geq 0 \text{ or } \sin x, \cos x \leq 0$$

$x$  must be in the first or third quadrant, i.e.  $k\pi \leq x \leq \pi + k\pi$  where  $k \in \mathbb{Z}$

Also,

$$\begin{aligned} 16 - x^2 &> 0 \\ -(x+4)(x-4) &> 0 \implies -2 < x < 2 \end{aligned}$$

The domain of the equation is the intersection of these two sets. Let's leave this for now and see if there are any solutions to measure against this domain.

Applying the zero product rule, we have that either

$$\sqrt{\sin x \cos x} = 0 \quad \text{or} \quad 1 - \log(16 - x^2) = 0$$

We solve each equation.

$$\begin{aligned} \sqrt{\sin x \cos x} &= 0 \\ \sin x &= 0 \quad \text{or} \quad \cos x = 0 \\ x &= k\pi \quad \text{or} \quad \frac{\pi}{2} + k\pi \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

The other equation:

$$\begin{aligned} 1 - \log(16 - x^2) &= 0 \\ 1 &= \log(16 - x^2) \\ 10 &= 16 - x^2 \\ x^2 &= 6 \\ x &= \pm\sqrt{6} \end{aligned}$$

Now we need to see which of the solutions falls into the domain. First for  $\pm\sqrt{6}$ : They both are clearly between  $-2$  and  $2$ .

Since  $\sqrt{6}$  is between  $2$  and  $3$ , it is also between  $0$  and  $\pi$  and so it is in the domain.

Since  $-\sqrt{6}$  is between  $-3$  and  $-2$ , it is between  $-\pi$  and  $0$  and so it is not in the domain. So, it is not a solution.

Now for the other group,  $x = \frac{k\pi}{2}$  where  $k \in \mathbb{Z}$ . The only solutions are the ones between  $-2$  and  $2$ .

| $k$              | $-3$      | $-2$     | $-1$      | $0$ | $1$      | $2$     |
|------------------|-----------|----------|-----------|-----|----------|---------|
| $\frac{k\pi}{2}$ | $-4.7124$ | $-3.141$ | $-1.5708$ | $0$ | $1.5708$ | $3.141$ |
|                  | nope      | nope     | yes       | yes | yes      | nope    |

So, there are four solutions:  $-\frac{\pi}{2}$ ,  $0$ ,  $\frac{\pi}{2}$ , and  $\sqrt{6}$ .

5. In a triangle, the sides around angle  $\alpha$  have lengths  $\sin \alpha$  and  $\cos \alpha$ . The side opposite angle  $\alpha$  has length  $1 - \sin \alpha$ . Compute the sides and angles in the triangle.

Solution: We state the Law of Cosines for this triangle:

$$\begin{aligned} (1 - \sin \alpha)^2 &= \sin^2 \alpha + \cos^2 \alpha - 2(\sin \alpha)(\cos \alpha) \cos \alpha \\ 1 - 2\sin \alpha + \sin^2 \alpha &= 1 - 2\sin \alpha \cos^2 \alpha \\ -2\sin \alpha + \sin^2 \alpha &= -2\sin \alpha \cos^2 \alpha \\ 0 &= -2\sin \alpha \cos^2 \alpha + 2\sin \alpha - \sin^2 \alpha \\ 0 &= \sin \alpha (-2\cos^2 \alpha + 2 - \sin \alpha) \\ 0 &= \sin \alpha (-2(1 - \sin^2 \alpha) + 2 - \sin \alpha) \\ 0 &= \sin \alpha (-2 + 2\sin^2 \alpha + 2 - \sin \alpha) \\ 0 &= \sin \alpha (2\sin^2 \alpha - \sin \alpha) \\ 0 &= \sin^2 \alpha (2\sin \alpha - 1) \end{aligned}$$

$$\sin \alpha = 0 \text{ or } \sin \alpha = \frac{1}{2}$$

$$\alpha = k\pi \text{ or } \alpha = \frac{\pi}{6} + 2k\pi \text{ or } \alpha = \frac{5\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z}$$

Since  $\alpha$  is an angle in a triangle, the only possible solution is  $\alpha = 30^\circ$  and thus  $\sin \alpha = \frac{1}{2}$  and  $\cos \alpha = \frac{\sqrt{3}}{2}$ . The three sides are:  $\frac{1}{2}$ ,  $\frac{1}{2}$ , and  $\frac{\sqrt{3}}{2}$ . The angles are  $30^\circ$ ,  $30^\circ$ , and  $120^\circ$ .

6. Find the value of the parameter  $p$  for which the vector that translates the graph of  $y = x^2 - 4px + 2$  into the parabola  $y = x^2 + 2px - 4$  has the shortest possible length.

Solution: Let's see where the vertices are.

$$y = x^2 - 4px + 2 = (x - 2p)^2 - 4p^2 + 2 \quad V_1 (2p, -4p^2 + 2)$$

$$y = x^2 + 2px - 4 = (x + p)^2 - p^2 - 4 \quad V_2 (-p, -p^2 - 4)$$

So the vector  $v$  points from  $V_1$  to  $V_2$ . Its length is

$$\begin{aligned} d(V_1, V_2) &= \sqrt{(2p - (-p))^2 + (-4p^2 + 2 - (-p^2 - 4))^2} = \sqrt{(3p)^2 + (-4p^2 + 2 + p^2 + 4)^2} \\ &= \sqrt{(3p)^2 + (-3p^2 + 6)^2} = 3\sqrt{p^2 + (-p^2 + 2)^2} = 3\sqrt{p^2 + p^4 - 4p^2 + 4} = \\ &= 3\sqrt{p^4 - 3p^2 + 4} = 3\sqrt{\left(p^2 - \frac{3}{2}\right)^2 - \frac{9}{4} + 4} = 3\sqrt{\left(p^2 - \frac{3}{2}\right)^2 + \frac{7}{4}} \end{aligned}$$

This expression has a minimum value when  $p^2 = \frac{3}{2}$  and also  $p = \pm\sqrt{\frac{3}{2}}$ .

7. Find all  $(x, y)$  that satisfy the given equation:

$$\tan\left(\frac{2y}{x}\right) + \cot\left(\frac{2y}{x}\right) + \sqrt{4x - x^2} + \sqrt{x^2 - 10x + 16} + \sqrt{11x - x^2 - 18} = 3x$$

Solution: Let us first discuss the domain of the equation. First, focusing on the radicals,

$$\begin{aligned} 4x - x^2 &\geq 0 \\ x(4 - x) &\geq 0 \implies 0 \leq x \leq 4 \end{aligned}$$

$$\begin{aligned} x^2 - 10x + 16 &\geq 0 \\ (x - 8)(x - 2) &\geq 0 \implies x \leq 2 \text{ or } x \geq 8 \end{aligned}$$

$$\begin{aligned} 11x - x^2 - 18 &\geq 0 \\ -x^2 + 11x - 18 &\geq 0 \\ -(x - 2)(x - 9) &\geq 0 \implies 2 \leq x \leq 9 \end{aligned}$$

If we take the intersection of these three sets, we are left with only  $x = 2$ . Let's solve for  $y$

$$\begin{aligned} \tan\left(\frac{2y}{x}\right) + \cot\left(\frac{2y}{x}\right) + \sqrt{4x - x^2} + \sqrt{x^2 - 10x + 16} + \sqrt{11x - x^2 - 18} &= 3x \\ \tan\left(\frac{2y}{2}\right) + \cot\left(\frac{2y}{2}\right) + \sqrt{4 \cdot 2 - 2^2} + \sqrt{2^2 - 10 \cdot 2 + 16} + \sqrt{11 \cdot 2 - 2^2 - 18} &= 3 \cdot 2 \end{aligned}$$

$$\begin{aligned}
 \tan y + \cot y + \sqrt{4} + \sqrt{0} + \sqrt{0} &= 6 \\
 \tan y + \cot y + 2 &= 6 \\
 \tan y + \cot y &= 4 \\
 \tan y + \frac{1}{\tan y} &= 4 \\
 \tan^2 y + 1 &= 4 \tan y \\
 \tan^2 y - 4 \tan y + 1 &= 0 \\
 (\tan y - 2)^2 - 3 &= 0 \\
 (\tan y - 2 + \sqrt{3})(\tan y - 2 - \sqrt{3}) &= 0
 \end{aligned}$$

$$\begin{aligned}
 \tan y_1 &= 2 - \sqrt{3} & \tan y_2 &= 2 + \sqrt{3} \\
 y_1 &= \frac{\pi}{12} + k\pi & y_2 &= \frac{5\pi}{12} + k\pi \quad \text{where } k \in \mathbb{Z}
 \end{aligned}$$

So the solutions are  $\left\{ (2, y) : y = \frac{\pi}{12} + k\pi \text{ or } y = \frac{5\pi}{12} + k\pi \text{ where } k \in \mathbb{Z} \right\}$

8. Solve the given equation for  $x$  over the real numbers.

$$\sqrt{\log_a \sqrt[4]{ax} + \log_x \sqrt[4]{ax}} + \sqrt{\log_a \sqrt[4]{\frac{x}{a}} + \log_x \sqrt[4]{\frac{a}{x}}} = a$$

Solution: Consider the domain first:  $a, x > 0$   $a, x \neq 1$

$$\begin{aligned}
 \sqrt{\log_a \sqrt[4]{ax} + \log_x \sqrt[4]{ax}} + \sqrt{\log_a \sqrt[4]{\frac{x}{a}} + \log_x \sqrt[4]{\frac{a}{x}}} &= a \\
 \sqrt{\frac{1}{4} \log_a (ax) + \frac{1}{4} \log_x (ax)} + \sqrt{\frac{1}{4} \log_a \left(\frac{x}{a}\right) + \frac{1}{4} \log_x \left(\frac{a}{x}\right)} &= a \\
 \frac{1}{2} \sqrt{\log_a a + \log_a x + \log_x a + \log_x x} + \frac{1}{2} \sqrt{\log_a x - \log_a a + \log_x a - \log_x x} &= a \\
 \frac{1}{2} \sqrt{1 + \log_a x + \log_x a + 1} + \frac{1}{2} \sqrt{\log_a x - 1 + \log_x a - 1} &= a \\
 \sqrt{\log_a x + \log_x a + 2} + \sqrt{\log_a x + \log_x a - 2} &= 2a
 \end{aligned}$$

Recall that  $\log_P Q$  and  $\log_Q P$  are reciprocals of each other.

$$\sqrt{\log_a x + \frac{1}{\log_a x} + 2} + \sqrt{\log_a x + \frac{1}{\log_a x} - 2} = 2a$$

Claim:  $\log_a x > 0$

Proof: Let  $m$  be any positive number. Then  $m + \frac{1}{m} \geq 2$  because

$$\begin{aligned}
 (m - 1)^2 &\geq 0 & \text{equality true only when } m &= 1 \\
 m^2 - 2m + 1 &\geq 0 \\
 m^2 + 1 &\geq 2m & m > 0 \\
 m + \frac{1}{m} &\geq 2
 \end{aligned}$$

And if  $m$  is a negative number, then  $m + \frac{1}{m} \leq -2$  and thus

$$m + \frac{1}{m} - 2 \leq -4$$

and thus

If we want  $\sqrt{\log_a x + \frac{1}{\log_a x} - 2}$  to be defined,  $m = \log_a x$  can not be a negative number. So,  $\log_a x > 0$ .

Notice that  $\left(R + \frac{1}{R}\right)^2 = R^2 + \frac{1}{R^2} + 2$  and  $\left(R - \frac{1}{R}\right)^2 = R^2 + \frac{1}{R^2} - 2$ .

Thus  $\left(\sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}}\right)^2 = \log_a x + \frac{1}{\log_a x} + 2$  and  $\left(\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}}\right)^2 = \log_a x + \frac{1}{\log_a x} - 2$ . Thus

$$\begin{aligned} \sqrt{\log_a x + \frac{1}{\log_a x} + 2} + \sqrt{\log_a x + \frac{1}{\log_a x} - 2} &= 2a \\ \sqrt{\left(\sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}}\right)^2} + \sqrt{\left(\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}}\right)^2} &= 2a \\ \left|\sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}}\right| + \left|\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}}\right| &= 2a \end{aligned}$$

The first expression,  $\sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}}$  is clearly non-negative so we can drop the absolute value sign.

$$\sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}} + \left|\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}}\right| = 2a$$

Case 1. If  $\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}} \geq 0$

$$\begin{aligned} \sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}} + \left|\sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}}\right| &= 2a \\ \sqrt{\log_a x} + \frac{1}{\sqrt{\log_a x}} + \sqrt{\log_a x} - \frac{1}{\sqrt{\log_a x}} &= 2a \\ 2\sqrt{\log_a x} &= 2a \\ \sqrt{\log_a x} &= a \\ \log_a x &= a^2 \\ x &= a^{(a^2)} \end{aligned}$$