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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve each of the following equations over the real numbers.

a) $(3x^2 + 2x - 8) \cdot \sqrt{x^2 + 2x - 3} = 0$

b) $\sqrt{3x^2 + 2x - 8} \cdot \sqrt{x^2 + 2x - 3} = 0$

Solution: We factor: $3x^2 + 2x - 8 = (3x - 4)(x + 2)$ and $x^2 + 2x - 3 = (x + 3)(x - 1)$.

a) The square root is defined if $x^2 + 2x - 3 \geq 0$, that is, $x \leq -3$ or $x \geq 1$. The product is zero if one of the factors is zero: $x = \frac{4}{3}$ or $x = -2$ from the first factor, and $x = -3$ or $x = 1$ from the second one. Since -2 is not in the domain, the solutions are $x = -3$, $x = 1$, and $x = \frac{4}{3}$.

b) Now both expressions under the square roots must be non-negative: $x \leq -2$ or $x \geq \frac{4}{3}$, and $x \leq -3$ or $x \geq 1$. Together: $x \leq -3$ or $x \geq \frac{4}{3}$. Of the four candidates, $x = -2$ and $x = 1$ are not in this domain (for example, if $x = 1$, then $3x^2 + 2x - 8 = -3 < 0$). So the solutions are $x = -3$ and $x = \frac{4}{3}$.

2. The diagonals of a parallelogram are 8 and 12 units long, and its area is 40 square units. Find the lengths of the sides of the parallelogram.

Solution: The diagonals of a parallelogram bisect each other, so they divide it into four triangles, each with sides 4 and 6 (half of the diagonals). If φ is the angle between the diagonals, then the area is

$$T = \frac{1}{2} \cdot 8 \cdot 12 \cdot \sin \varphi = 48 \sin \varphi = 40 \quad \text{so} \quad \sin \varphi = \frac{5}{6} \quad \text{and} \quad \cos \varphi = \pm \frac{\sqrt{11}}{6}$$

The two sides are opposite the angles φ and $180^\circ - \varphi$ in these triangles. By the Law of Cosines,

$$a^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \frac{\sqrt{11}}{6} = 52 - 8\sqrt{11}, \quad a \approx 5.05$$

$$b^2 = 4^2 + 6^2 + 2 \cdot 4 \cdot 6 \cdot \frac{\sqrt{11}}{6} = 52 + 8\sqrt{11}, \quad b \approx 8.86$$

The sides are $\sqrt{52 - 8\sqrt{11}} \approx 5.05$ and $\sqrt{52 + 8\sqrt{11}} \approx 8.86$ units.

3. The sum of the first four terms of the geometric sequence (a_n) is 16, and $2(a_4 - a_1) = 7(a_3 - a_2)$. Find the first term and the common ratio of the sequence.

Solution: Let a be the first term and q the common ratio ($a \neq 0$). The second condition is

$$2a(q^3 - 1) = 7a(q^2 - q), \quad \text{that is,} \quad 2(q - 1)(q^2 + q + 1) = 7q(q - 1)$$

If $q = 1$, this holds, and the sum is $4a = 16$, so $a = 4$: the constant sequence $4, 4, 4, 4, \dots$ is a solution.

If $q \neq 1$, we divide by $q - 1$: $2q^2 + 2q + 2 = 7q$, that is, $2q^2 - 5q + 2 = 0$, so $q = 2$ or $q = \frac{1}{2}$.

If $q = 2$: $a(1 + 2 + 4 + 8) = 15a = 16$, so $a = \frac{16}{15}$.

If $q = \frac{1}{2}$: $a\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8}a = 16$, so $a = \frac{128}{15}$.

So $a = 4, q = 1$; or $a = \frac{16}{15}, q = 2$; or $a = \frac{128}{15}, q = \frac{1}{2}$.

4. Solve the given system of trigonometric equations over the real numbers.

$$\begin{aligned} \sin(x - y) &= 2 \sin x \sin y \\ x + y &= \frac{\pi}{2} \end{aligned}$$

Solution: From the second equation, $y = \frac{\pi}{2} - x$, so $\sin y = \cos x$ and

$$\sin(x - y) = \sin\left(2x - \frac{\pi}{2}\right) = -\cos 2x$$

The first equation becomes $-\cos 2x = 2 \sin x \cos x = \sin 2x$. If $\cos 2x = 0$, then also $\sin 2x = 0$, which is impossible; so we can divide by $\cos 2x$: $\tan 2x = -1$. So $2x = -\frac{\pi}{4} + k\pi$, and the solutions are

$$x = -\frac{\pi}{8} + k \cdot \frac{\pi}{2}, \quad y = \frac{\pi}{2} - x = \frac{5\pi}{8} - k \cdot \frac{\pi}{2}$$

where k is any integer.

5. Find the equation of the chord of the parabola $y = x^2$ whose midpoint is $P(1, 6)$.

Solution: Let the endpoints of the chord be (x_1, x_1^2) and (x_2, x_2^2) . Since the midpoint is $P(1, 6)$, $x_1 + x_2 = 2$. The slope of the chord is

$$\frac{x_2^2 - x_1^2}{x_2 - x_1} = x_1 + x_2 = 2$$

So the chord is on the line through P with slope 2: $y = 2x + 4$. (Check: the line intersects the parabola where $x^2 = 2x + 4$, that is, $x = 1 \pm \sqrt{5}$; the average of these is 1, and the average of the y -coordinates is $\frac{(1 + \sqrt{5})^2 + (1 - \sqrt{5})^2}{2} = 6$.)

6. Among the rectangles with perimeter 60 units, find the area of the one whose diagonals are as short as possible.

Solution: Let the sides be a and b ; then $a + b = 30$. The square of the diagonal is

$$a^2 + b^2 = (a + b)^2 - 2ab = 900 - 2ab$$

This is smallest when the area ab is largest. By the inequality between the arithmetic and geometric means, $ab \leq \left(\frac{a+b}{2}\right)^2 = 225$, with equality if and only if $a = b = 15$. So the rectangle with the shortest diagonals is the square with side 15, and its area is 225 square units.

7. Solve the given inequality, where a is a real parameter.

$$\log_a (x - a) > \log_{\frac{1}{a}} (x + a)$$

Solution: For the logarithms, we need $a > 0$, $a \neq 1$, $x - a > 0$, and $x + a > 0$; since $a > 0$, this means $x > a$. Since $\log_{\frac{1}{a}} t = -\log_a t$, the inequality is

$$\log_a (x - a) + \log_a (x + a) > 0, \quad \text{that is,} \quad \log_a (x^2 - a^2) > 0$$

If $a > 1$, the logarithm is increasing, so we need $x^2 - a^2 > 1$, that is, $x > \sqrt{a^2 + 1}$ (since $x > a > 0$).

If $0 < a < 1$, the logarithm is decreasing, so we need $0 < x^2 - a^2 < 1$, that is, $a < x < \sqrt{a^2 + 1}$.

(If $a \leq 0$ or $a = 1$, the inequality has no meaning.)

8. Let a , b , and c be the sides of a triangle, and α , β , and γ the opposite angles, respectively. Prove that

$$\begin{aligned} a^2 + b^2 - ab\sqrt{2} \cos(\gamma + 45^\circ) &= b^2 + c^2 - bc\sqrt{2} \cos(\alpha + 45^\circ) \\ &= c^2 + a^2 - ca\sqrt{2} \cos(\beta + 45^\circ) \end{aligned}$$

Solution: Since $\sqrt{2} \cos(\gamma + 45^\circ) = \sqrt{2} \left(\cos \gamma \cdot \frac{\sqrt{2}}{2} - \sin \gamma \cdot \frac{\sqrt{2}}{2} \right) = \cos \gamma - \sin \gamma$,

$$a^2 + b^2 - ab\sqrt{2} \cos(\gamma + 45^\circ) = a^2 + b^2 - ab \cos \gamma + ab \sin \gamma$$

By the Law of Cosines, $ab \cos \gamma = \frac{a^2 + b^2 - c^2}{2}$, and $ab \sin \gamma = 2T$, where T is the area of the triangle. So the expression is

$$a^2 + b^2 - \frac{a^2 + b^2 - c^2}{2} + 2T = \frac{a^2 + b^2 + c^2}{2} + 2T$$

This is symmetric in a , b , and c , so the other two expressions (which we get by renaming the sides and angles) are equal to the same value.