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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The perimeter of a right triangle is 40 cm, its area is 60 cm^2 . How long are its sides?

Solution: Let a , b , and c denote the sides where c is the hypotenuse.

$$a^2 + b^2 = c^2$$

$$a + b + c = 40$$

$$\frac{1}{2}ab = 60$$

$$a^2 + b^2 = c^2$$

$$c = 40 - a - b$$

$$ab = 120$$

$$a^2 + b^2 = (40 - a - b)^2$$

$$ab = 120$$

$$a^2 + b^2 = 1600 + a^2 + b^2 - 80a - 80b + 2ab$$

$$0 = 1600 - 80a - 80b + 2ab$$

$$0 = 800 - 40a - 40b + ab$$

$$0 = 800 - 40a - 40b + 120$$

$$40(a + b) = 920$$

$$a + b = 23$$

$$a + b = 23$$

$$ab = 120$$

$$a(23 - a) = 120$$

$$-a^2 + 23a - 120 = 0$$

$$a^2 - 23a + 120 = 0$$

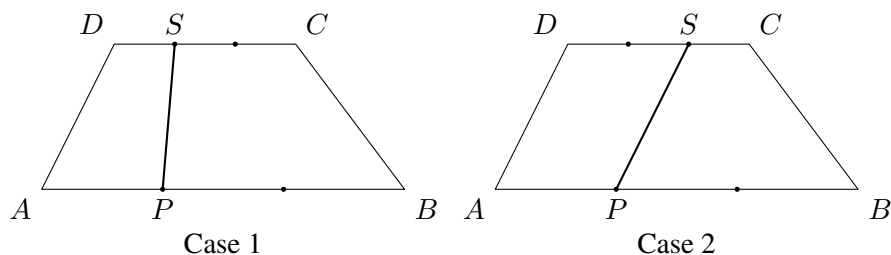
$$(a - 8)(a - 15) = 0$$

$$a_1 = 8 \quad a_2 = 15$$

If $a = 8$ then $b = 15$ and if $a = 15$ then $b = 8$. So the two solutions are really the same triangle with legs 8 cm and 15 cm. The hypotenuse is then $\sqrt{8^2 + 15^2} = 17$ cm.

2. Suppose that $ABCD$ is a trapezoid with parallel sides AB and CD . Let P and Q be points on AB that divide AB into three equal line segments. Let S and T be points on CD that divide CD into three equal line segments. We connect P and S . The ratio between the two regions created by PS is 2 to 3. What is the ratio of AB to CD ?

Solution:



The problem is not exactly clear. There are two possibilities and one of them is easily eliminated.

Suppose $AB = 3x$ and $CD = 3y$. Let h denote the height of the trapezoid.

Case 1. One piece is bounded by x and y and the other is by $2x$ and $2y$

Total area: $\frac{3x + 3y}{2}h$

Area of the first part: $\frac{x + y}{2}h$ Area of second part: $\frac{2x + 2y}{2}h$

The ratio between these two quantities is

$$\frac{A_1}{A_2} = \frac{\frac{x + y}{2}h}{\frac{2x + 2y}{2}h} = \frac{1}{2}$$

so this is impossible.

Case 2. One piece is bounded by $2x$ and y and the other is by x and $2y$

Total area: $\frac{3x + 3y}{2}h$

Area of the first part: $\frac{2x + y}{2}h$ Area of second part: $\frac{x + 2y}{2}h$

The ratio between these two quantities is

$$\frac{A_1}{A_2} = \frac{\frac{2x + y}{2}h}{\frac{x + 2y}{2}h} = \frac{2x + y}{x + 2y} = \frac{2}{3}$$

We can now solve for the ratio between x and y

$$\begin{aligned} \frac{2x + y}{x + 2y} &= \frac{2}{3} \\ 6x + 3y &= 2x + 4y \\ 4x &= y \\ \frac{4}{1} &= \frac{y}{x} \end{aligned}$$

So $\frac{CD}{AB} = \frac{3y}{3x} = \frac{y}{x} = \frac{4}{1}$, and so $AB : CD = 1 : 4$. (If we label the two regions the other way, so that the ratio of their areas is $3 : 2$, we get $AB : CD = 4 : 1$. Either way, one of the parallel sides is four times as long as the other.)

3. Solve the given equation.

$$\cos x + 2 \tan x = \frac{7}{4 \cos x}$$

Solution: domain: $\cos x \neq 0 \implies x \neq \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$

$$\begin{aligned} \cos x + 2 \tan x &= \frac{7}{4 \cos x} \\ \cos x + \frac{2 \sin x}{\cos x} &= \frac{7}{4 \cos x} && \text{multiply by } 4 \cos x \\ 4 \cos^2 x + 8 \sin x &= 7 \\ 4(1 - \sin^2 x) + 8 \sin x - 7 &= 0 \\ 4 - 4 \sin^2 x + 8 \sin x - 7 &= 0 \\ -4 \sin^2 x + 8 \sin x - 3 &= 0 \\ 4 \sin^2 x - 8 \sin x + 3 &= 0 \\ (2 \sin x - 1)(2 \sin x - 3) &= 0 \end{aligned}$$

$$\begin{aligned} \sin x_1 &= \frac{1}{2} & \sin x_2 &= \frac{3}{2} \\ \sin x_1 = \frac{1}{2} &\implies x = \frac{\pi}{6} + 2k\pi \text{ or } x = \frac{5\pi}{6} + 2k\pi \text{ where } k \in \mathbb{Z} \\ \sin x_2 &= \frac{3}{2} \implies \text{no solution here} \end{aligned}$$

So the solution is $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$.

4. The sum of the first six terms in a geometric sequence is three times as much as the sum of the first three terms. What is the common quotient in the sequence? (Assume that none of the terms is zero.)

Solution:

$$s_6 = 3s_3$$

Case 1. Suppose $q = 1$. Then $s_6 = 6a$ and $s_3 = 3a$

$$\begin{aligned} 6a &= 3(3a) \\ 6a &= 9a \\ 0 &= 3a \\ a &= 0 \end{aligned}$$

That is not possible.

Case 2. Suppose $q \neq 1$

$$\begin{aligned} a \frac{q^6 - 1}{q - 1} &= 3a \frac{q^3 - 1}{q - 1} && a \neq 0, q \neq 1 \\ q^6 - 1 &= 3(q^3 - 1) \\ q^6 - 1 &= 3q^3 - 3 \\ q^6 - 3q^3 + 2 &= 0 \\ (q^3 - 1)(q^3 - 2) &= 0 \\ q^3 = 1 &\implies q = 1, \text{ but } q \neq 1 \\ q^3 = 2 &\implies q = \sqrt[3]{2} \end{aligned}$$

So the quotient is $\sqrt[3]{2}$.

5. Solve the equation.

$$\log_8 (27^{x-1} + 37) = \log_2 (3^{x-1} + 1)$$

Solution:

$$\begin{aligned} \log_8 (27^{x-1} + 37) &= \log_2 (3^{x-1} + 1) \\ \frac{\log_2 (27^{x-1} + 37)}{\log_2 8} &= \log_2 (3^{x-1} + 1) \\ \frac{\log_2 (27^{x-1} + 37)}{3} &= \log_2 (3^{x-1} + 1) \\ \log_2 \left(\sqrt[3]{27^{x-1} + 37} \right) &= \log_2 (3^{x-1} + 1) \end{aligned}$$

The function $f(x) = \log_2 x$ is one-to-one

$$\begin{aligned} \sqrt[3]{27^{x-1} + 37} &= 3^{x-1} + 1 \\ 27^{x-1} + 37 &= (3^{x-1} + 1)^3 \\ 27^{x-1} + 37 &= (3^{x-1})^3 + 3(3^{x-1})^2 + 3(3^{x-1}) + 1 \\ 27^{x-1} + 36 &= 27^{x-1} + 3(3^{x-1})^2 + 3(3^{x-1}) \\ 36 &= 3(3^{x-1})^2 + 3(3^{x-1}) \\ 12 &= (3^{x-1})^2 + 3^{x-1} \\ 0 &= (3^{x-1})^2 + 3^{x-1} - 12 \\ 0 &= (3^{x-1} + 4)(3^{x-1} - 3) \\ 3^{x-1} &= -4 \quad \text{or} \quad 3^{x-1} = 3 \end{aligned}$$

There is no solution from the first one.

$$\begin{aligned} 3^{x-1} &= 3 \\ x - 1 &= 1 \\ x &= 2 \end{aligned}$$

We check: if $x = 2$, then $\log_8 (27 + 37) = \log_8 64 = 2$ and $\log_2 (3 + 1) = \log_2 4 = 2$. So $x = 2$ is the solution.

6. Find the equation for all circles that are tangent to the line $x = 2$, contain the origin, and intersects the line $y = x$ in a chord of length of $\sqrt{2}$.

Solution: Let (k, h) be the center and r the radius of such a circle. Since the circle is tangent to the line $x = 2$, the distance between the center and this line is the radius: $r = |k - 2|$. Since the circle contains the origin,

$$\begin{aligned} k^2 + h^2 &= r^2 = (k - 2)^2 \\ k^2 + h^2 &= k^2 - 4k + 4 \\ k &= 1 - \frac{h^2}{4} \end{aligned}$$

In particular, $k \leq 1 < 2$, and so $r = 2 - k = 1 + \frac{h^2}{4}$.

Next, we intersect the circle with the line $y = x$:

$$\begin{aligned} (x - k)^2 + (x - h)^2 &= r^2 \\ 2x^2 - 2(k + h)x + k^2 + h^2 &= r^2 \quad \text{recall } k^2 + h^2 = r^2 \\ 2x^2 - 2(k + h)x &= 0 \\ 2x(x - (k + h)) &= 0 \end{aligned}$$

So the two intersection points are the origin and $(k+h, k+h)$. The length of the chord between them is $\sqrt{2}|k+h|$, and this must be $\sqrt{2}$. So $|k+h| = 1$.

Case 1. $k+h = 1$

$$\begin{aligned} 1 - \frac{h^2}{4} + h &= 1 \\ h(4-h) &= 0 \\ h &= 0 \quad \text{or} \quad h = 4 \end{aligned}$$

If $h = 0$, then $k = 1$ and $r = 1$. If $h = 4$, then $k = -3$ and $r = 5$.

Case 2. $k+h = -1$

$$\begin{aligned} 1 - \frac{h^2}{4} + h &= -1 && \text{multiply by } -4 \\ h^2 - 4h - 8 &= 0 \\ h &= 2 \pm 2\sqrt{3} \end{aligned}$$

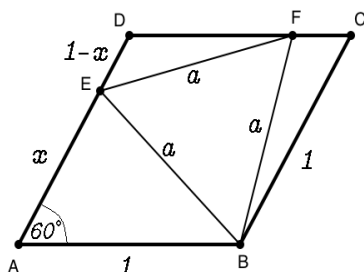
Then $k = -1 - h = -3 \mp 2\sqrt{3}$ and $r = 2 - k = 5 \pm 2\sqrt{3}$.

So there are four such circles:

$$\begin{aligned} (x-1)^2 + y^2 &= 1 \\ (x+3)^2 + (y-4)^2 &= 25 \\ (x+3+2\sqrt{3})^2 + (y-2-2\sqrt{3})^2 &= (5+2\sqrt{3})^2 = 37+20\sqrt{3} \\ (x+3-2\sqrt{3})^2 + (y-2+2\sqrt{3})^2 &= (5-2\sqrt{3})^2 = 37-20\sqrt{3} \end{aligned}$$

7. The angle at vertex A in a rhombus measures 120° . We draw the equilateral triangles that have one vertex at A and the two other vertices on the sides opposite A . Of these triangles, which one will have the smallest area? What is the ratio between this smallest area and the area of the rhombus?

Solution: Let us assume that the rhombus has sides 1 unit long. (In the picture, we labeled the rhombus so that the 120° angle is at vertex B .)



An equilateral triangle's area, in terms of its side a is $A = \frac{1}{2}a^2 \sin 60^\circ = \frac{\sqrt{3}}{4}a^2$. For minimal area, we need a minimal side. Apply the Law of Cosines to triangle ABE

$$\begin{aligned} a^2 &= 1^2 + x^2 - 2 \cdot 1 \cdot x \cdot \cos 60^\circ \\ &= x^2 + 1 - 2x \cdot \frac{1}{2} \\ &= x^2 - x + 1 \\ &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

The minimum value of a^2 occurs when $x = \frac{1}{2}$. Then $a = \frac{\sqrt{3}}{2}$, and so the area of this triangle is

$$A_{\text{tr}} = \frac{\sqrt{3}}{4} \cdot \frac{3}{4} = \frac{3\sqrt{3}}{16} \text{ unit}^2$$

The area of the rhombus is twice the area of triangle ABD , which is

$$A_{\text{r}} = 2 \left(\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 60^\circ \right) = \frac{\sqrt{3}}{2} \text{ unit}^2$$

The ratio between the two areas is

$$\frac{A_{\text{tr}}}{A_{\text{r}}} = \frac{\frac{3\sqrt{3}}{16}}{\frac{\sqrt{3}}{2}} = \frac{3}{16} \cdot 2 = \frac{3}{8}$$

8. Consider an arithmetic sequence consisting of different terms. The last term of the sequence is nine times the first term; the sum of the last three terms is five times the sum of the first three terms. How many terms are in the sequence?

Solution: Let a , d , n represent the first term, the common difference, and the number of terms. Also note that $d \neq 0$, since the terms are different.

$$\begin{aligned} a_n &= 9a \\ a_{n-2} + a_{n-1} + a_n &= 5(a_1 + a_2 + a_3) \end{aligned}$$

$$\begin{aligned} a + (n-1)d &= 9a \\ (n-1)d &= 8a \end{aligned}$$

These are two equations for three unknowns - but we only need to find n .

$$\begin{aligned} a_{n-2} + a_{n-1} + a_n &= 5(a_1 + a_2 + a_3) \\ a + (n-3)d + a + (n-2)d + a + (n-1)d &= 5(a + a + d + a + 2d) \\ 3a + (3n-6)d &= 5(3a + 3d) \\ a + (n-2)d &= 5a + 5d \\ (n-2)d &= 4a + 5d \\ nd - 2d &= 4a + 5d \\ nd &= 4a + 7d & 8a = (n-1)d \\ nd &= \frac{(n-1)d}{2} + 7d \\ 2nd &= (n-1)d + 14d \\ 2nd &= nd - d + 14d \\ nd &= 13d & d \neq 0 \\ n &= 13 \end{aligned}$$

So the sequence has 13 terms.