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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Write the equation of the circle that contains the points $A\,(1,4)$ and $B\,(5,0)$ and whose center is on the line $x+y=3$.

   Solution: the center is also contained on the perpendicular bisector of line segment $AB$. So, if we find the perpendicular bisector and intersect is with the given line, the intersection point will be the center.

   $$m_{AB}=\frac{0-4}{5-1}=-1 \qquad \text{perpendiculer to that: } m_p=1$$

   midpoint:

   $$x_M=\frac{x_A+x_B}{2}=3 \qquad y_M=\frac{y_A+y_B}{2}=2 \qquad M\,(3,2)$$

   So the line is $y-2=1\,(x-3)$ or $y=x-1$. We intersect this with the line given:

   $$\begin{aligned} x+y &= 3 \\ y &= x-1 \end{aligned}$$

   $$\begin{aligned} x+x-1 &= 3 \\ 2x &= 4 \\ x &= 2 \implies y=1 \end{aligned}$$

   and so the center of the circle is $(2,1)$. The radius is the distance of this point from point $A$.

   $$r=\sqrt{(2-1)^2+(1-4)^2}=\sqrt{1+9}=\sqrt{10}$$

   So the equation of the circle is $(x-2)^2+(y-1)^2=10$. We leave checking for the reader.

2. Solve the equation over the real numbers:

   $$\sqrt{3-x}+\frac{6}{\sqrt{3-x}}=\sqrt{9-5x}$$

Solution: $x \neq 3$. Also, $x < 3$ and $x \leq \frac{5}{9}$ So, $x \leq \frac{5}{9}$

$$\begin{aligned}
\sqrt{3-x} + \frac{6}{\sqrt{3-x}} &= \sqrt{9-5x} \\
3 - x + 6 &= \sqrt{9-5x}\sqrt{3-x} \\
9 - x &= \sqrt{5x^2 - 9x - 15x + 27} \\
9 - x &= \sqrt{5x^2 - 24x + 27} \qquad \text{square} \\
x^2 - 18x + 81 &= 5x^2 - 24x + 27 \\
0 &= 4x^2 - 6x - 54 \\
0 &= 2x^2 - 3x - 27 \\
0 &= (x+3)(2x-9)
\end{aligned}$$

$$x_1 = -3 \quad x_2 = \frac{9}{2}$$

We check $x = -3$

$$\begin{aligned}
\text{LHS} &= \sqrt{3-(-3)} + \frac{6}{\sqrt{3-(-3)}} = \sqrt{6} + \frac{6}{\sqrt{6}} = 2\sqrt{6} \\
\text{RHS} &= \sqrt{9-5(-3)} = \sqrt{24} = 2\sqrt{6}
\end{aligned}$$

and if $x = \frac{9}{2}$, then $\sqrt{9-5x} = \sqrt{9 - \frac{45}{2}} = \sqrt{-\frac{27}{2}}$ is undefined. So the only solution is $-3$.

3. Solve the equation over the real numbers.

$$2\log(x+3) - \log x = \log(25x+3) - 1$$

Solution: domain: $x > -3$ and $x > -\frac{3}{25}$ and so $x > -3$

$$\begin{aligned}
2\log(x+3) - \log x &= \log(25x+3) - 1 \\
\log(x+3)^2 - \log x &= \log(25x+3) - \log 10 \\
\log\frac{(x+3)^2}{x} &= \log\frac{(25x+3)}{10}
\end{aligned}$$

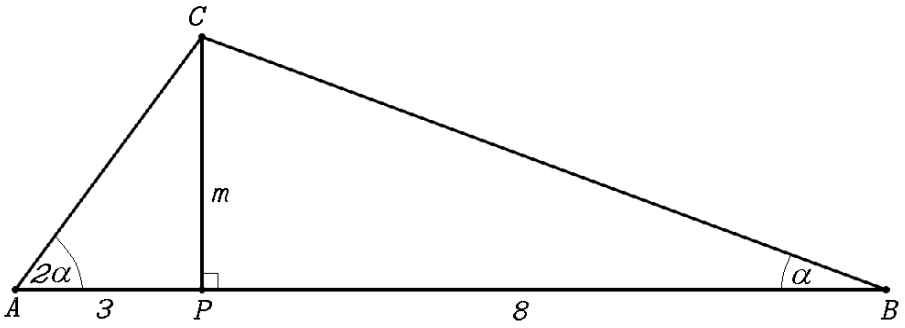
The function $f(x) = \log x$ is one-to-one

$$\begin{aligned}
\frac{(x+3)^2}{x} &= \frac{(25x+3)}{10} \\
10(x+3)^2 &= x(25x+3) \\
10\left(x^2 + 6x + 9\right) &= 25x^2 + 3x \\
10x^2 + 60x + 90 &= 25x^2 + 3x \\
0 &= 15x^2 - 57x - 90 \\
0 &= 5x^2 - 19x - 30
\end{aligned}$$

$$x_{1,2} = \frac{19 \pm \sqrt{19^2 - 4 \cdot 5(-30)}}{2 \cdot 5} = \frac{19 \pm \sqrt{361+600}}{10} = \frac{19 \pm \sqrt{961}}{10} = \frac{19 \pm 31}{10} = \begin{cases} \frac{19+31}{10} = \frac{50}{10} = 5 \\ \frac{19 \pm 31}{10} = \frac{-12}{10} = -\frac{6}{5} \end{cases}$$

4. Consider triangle $ABC$. The altitude (or height) belonging to side $AB$ splits the side into line segments of lengths $8$ and $3$ units long. The angle of the triangle near the $3$ unit long segment is twice the angle near the $8$ unit long segment. Find the triangle's sides and area.

   Solution: Let $m$ denote the length of the height. Clearly $m > 0$.




$$\begin{aligned}
\tan\alpha &= \frac{m}{8} \\
\tan 2\alpha &= \frac{m}{3}
\end{aligned}$$

$$\begin{aligned}
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\
\frac{m}{3} &= \frac{2\left(\frac{m}{8}\right)}{1-\left(\frac{m}{8}\right)^2} \\
\frac{m}{3} &= \frac{\frac{m}{4}}{1-\frac{m^2}{64}} \\
\frac{m}{3} &= \frac{\frac{m}{4}}{\frac{64-m^2}{64}} \\
\frac{m}{3} &= \frac{m}{4}\cdot\frac{64}{64-m^2} \qquad m>0 \\
\frac{1}{3} &= \frac{16}{64-m^2} \\
64-m^2 &= 48 \\
16 &= m^2 \\
m &= \pm 4 \implies m = 4
\end{aligned}$$

We can now find the other two sides via the Pythagorean Theorem.

$$\begin{aligned}
AC &= \sqrt{m^2+3^2} = \sqrt{25} = 5 \\
BC &= \sqrt{m^2+8^2} = \sqrt{80} = 4\sqrt{5}
\end{aligned}$$

The area of the triangle is $\frac{1}{2}(11)\,4 = 22$ unit$^2$.

5. A square based straight pyramid was cut into two by a plane parallel to its base and containing the midpoint of its main altitude. The volume of the pyramid created is $336\,\mathrm{cm}^3$, and its height is one third of the side of the base of the original pyramid. Find the other part's volume. (That is the part that is not a pyramid, also called a frustum.) Find the angle that is formed between the side face and the base of the pyramid.

   Solution: Let $H$ and $h$ denote the height of the two pyramids, and $A$ and $a$ the sides of the square bases, where $h < H$ and $A < A$. The height was cut into a half. That means that the original pyramid and the smaller pyramid are similar with a magnifying factor of 2.

$$2 = \frac{H}{h} = \frac{A}{a}$$

   This imples that $A = 2a$ and $H = 2h$ and that the volumes of the two part have a ratio of $1 : 8$.

$$\begin{aligned} V_{\text{small}} &= \frac{1}{3}a^2h \\ V_{\text{large}} &= \frac{1}{3}A^2H = \frac{1}{3}(2a)^2(2h) = 8 \cdot \frac{1}{3}a^2h = 8 \cdot V_{\text{small}} \end{aligned}$$

   So the other part's volume is $8 \cdot 336\,\mathrm{cm}^3 = 2688\,\mathrm{cm}^3$

   For the angle formed between a triangular face and the base:

$$\tan\alpha = \frac{H}{\frac{A}{2}} = \frac{2H}{A} = \frac{2h}{a}$$

   Clearly $h = \dfrac{H}{2}$ and $\dfrac{h}{H} = \dfrac{a}{A}$ implies that also $a = \dfrac{A}{2}$. The volume of the smaller pyramid is 336

$$\begin{aligned} \frac{1}{3}a^2h &= 336 \\ a^2h &= 1008 \end{aligned}$$

   The height of the smaller pyramid is one third of the side of the base of the original pyramid:

$$h = \frac{A}{3} = \frac{2a}{3} \quad \Longrightarrow \quad \frac{h}{a} = \frac{2}{3}$$

$$\tan\alpha = \frac{H}{\frac{A}{2}} = \frac{2H}{A} = \frac{2h}{a} = 2\left(\frac{2}{3}\right) = \frac{4}{3}$$

   and so $\alpha = \tan^{-1}\left(\dfrac{4}{3}\right) \approx 53.130102^\circ$. Notice that we never needed to compute the values of $a$ and $h$.

6. The first term of an arithmetic sequence equals to the common ratio of a geometric sequence. The first term of this geometric sequence is the difference of the arithmetic sequence. The sum of the first five terms of the arithmetic sequence is 40, the sum of the first two terms of the geometric sequence is 10. Find these sequences.

   Solution: Let $a$ denote the first term of the arithmetic sequence and $d$ its common difference. Then

$$\begin{aligned} s_5 &= 40 \\ 40 &= \frac{2a + 4d}{2}(5) \\ 8 &= a + 2d \qquad a = 8 - 2d \end{aligned}$$

The geometric sequence's first term is $d$ and its second term is $ad$. So the sum of the first two terms in the geometric sequence is

$$\begin{aligned} d + ad &= 10 \\ d + (8 - 2d)\, d &= 10 \\ d + 8d - 2d^2 &= 10 \\ 0 &= 2d^2 - 9d + 10 \\ 0 &= (2d - 5)\,(d - 2) \end{aligned}$$

$$d_1 = \frac{5}{2} \quad d_2 = 2$$

If $d = \frac{5}{2}$, then $a = 8 - 2d = 3$, and if $d = 2$, then $a = 4$.

For the first solution, the arithmetic sequence is $a = 3$ and $d = \frac{5}{2}$

$\frac{6}{2}, \frac{11}{2}, \frac{16}{2}, \frac{21}{2}, \frac{26}{2}$ or

$$3, \frac{11}{2}, 8, \frac{21}{2}, 13$$

the sum is indeed $24 + 16 = 40$ and the geometric sequence is $a = \frac{5}{2}$ and $r = 3$

$$\frac{5}{2}, \frac{15}{2}$$

- the sum is indeed $10$ so this is a solution.

For the second solution, the arithmetic sequence is $a = 4$ and $d = 2$

Then the arithmetic sequence is $4, 6, 8, 10, 12$ and the sum is $40$ and the geometric sequence is $2, 8$ the sum is indeed $10$. Both answers work.

$$\text{arithmetic: } 3, \frac{11}{2}, 8, \frac{21}{2}, 13 \quad \text{gemoetric: } \frac{5}{2}, \frac{15}{2} \quad (a = 3, d = \frac{5}{2})$$

$$\text{arithmetic: } 4, 6, 8, 10, 12 \quad \text{gemoetric: } 2, 8 \quad (a = 4, d = 2)$$

7. Solve the following equation over the real numbers.

$$\frac{1 + \tan x}{1 - \tan x} = 1 + \sin 2x$$

Solution: domain: $\tan x$ exists and so $x \neq \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$ and $1 - \tan x \neq 0$ and so $x \neq \frac{\pi}{4} + k\pi$ where $k \in \mathbb{Z}$.

$$\begin{aligned}
\frac{1+\tan x}{1-\tan x} &= 1 + \sin 2x \\
\frac{1+\dfrac{\sin x}{\cos x}}{1-\dfrac{\sin x}{\cos x}} &= 1 + 2\sin x \cos x \\
\frac{\cos x + \sin x}{\cos x - \sin x} &= 1 + 2\sin x \cos x \\
\cos x + \sin x &= (1 + 2\sin x \cos x)(\cos x - \sin x) \\
0 &= (1 + 2\sin x \cos x)(\cos x - \sin x) - (\sin x + \cos x) \\
0 &= \left(\sin^2 x + \cos^2 x + 2\sin x \cos x\right)(\cos x - \sin x) - (\sin x + \cos x) \\
0 &= (\sin x + \cos x)^2 (\cos x - \sin x) - (\sin x + \cos x) \\
0 &= (\sin x + \cos x)\left[(\cos x + \sin x)(\cos x - \sin x) - 1\right] \\
0 &= (\sin x + \cos x)\left(\cos^2 x - \sin^2 x - 1\right) \\
0 &= (\sin x + \cos x)\left(\cos^2 x - 1 - \sin^2 x\right) \\
0 &= (\sin x + \cos x)\left(-\sin^2 x - \sin^2 x\right) \\
0 &= -2\sin^2 x (\sin x + \cos x)
\end{aligned}$$

$$\sin^2 x = 0 \implies \sin x = 0 \implies x = k\pi \text{ where } k \in \mathbb{Z}$$

$$\begin{aligned}
\sin x + \cos x &= 0 \\
\sin x &= -\cos x \qquad \cos x \neq 0 \\
\tan x &= -1 \implies x = \frac{3\pi}{4} + k\pi \text{ where } k \in \mathbb{Z}
\end{aligned}$$

8. (In Hungary at the time, the lottery rules were: we selected five out of the set $\{1, 2, 3, ...90\}$.) A person wants to select the numbers to be played on the lottery as follows: after selecting the smallest two numbers, the third number would be the sum of the first two; the fourth number would be the sum of the first three numbers, and the fifth number would be the sum of the first four.

a) What is the greatest possible value of the smallest number in the set of five?

Solution: let $x$ denote this first number. Then the next one is at least $x+1$. Then the third number is $x+x+1 = 2x+1$. Then the fourth number is $x + x + 1 + 2x + 1 = 4x + 2$ and finally, the fifth number is $x + x + 1 + 2x + 1 + 4x + 2 = 8x + 4$. This last number must be 90 or less.

$$\begin{aligned}
8x + 4 &\leq 90 \\
8x &\leq 86 \\
x &\leq \frac{86}{8}
\end{aligned}$$

So the greatest possible value would be 10. Let's check. If the first number is 10 and the second one is 11, then the third is 21, the fourth is $10 + 11 + 21 = 42$, and the fifth is $10 + 11 + 21 + 42 = 84$. But if the first number is 11, then the second one is 12, then the five numbers, similarly, are $11, 12, 23, 46, 92$ and that would be impossible.

b) If we select the smallest number to be the greatest possible, then what numbers would be playing?

Solution: Suppose the smallest number is 10. Then the second could be 11 and so we'd play

$$10, 11, 21, 42, 84$$

But if the second number we played was 12, we had

$$10, 12, 22, 44, 88$$

and it is pretty clear that 13 is not an option for the second number, so there are only these two possibilities.

c) In how many different ways can the numbers be played as described?

Solution: two different ways

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