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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1986. Problems 6/a, 7/a, and 8/a were recommended for students coming from vocational secondary schools (in place of problems 6, 7, and 8).

1. Find n if the ratio of the sum of the first n positive even numbers and the sum of the first n positive odd numbers is $\frac{21}{20}$.

   Solution: The sum of the first n positive even numbers is $2+4+\ldots+2n = n(n+1)$, and the sum of the first n positive odd numbers is $1+3+\ldots+(2n-1) = n^2$. So

   $$\frac{n(n+1)}{n^2} = \frac{n+1}{n} = \frac{21}{20}, \qquad 20n+20 = 21n, \qquad n = 20$$

2. Solve the given system of equations over the real numbers.

   $$\begin{aligned} 8^{2x+1} &= 32 \cdot 2^{4y-1} \\ 5 \cdot 5^{x-y} &= \sqrt{25^{2y+1}} \end{aligned}$$

   Solution: We write both equations with a common base. The first one is $2^{6x+3} = 2^5 \cdot 2^{4y-1} = 2^{4y+4}$, so $6x+3 = 4y+4$. The second one is $5^{1+x-y} = \left(5^{4y+2}\right)^{\frac{1}{2}} = 5^{2y+1}$, so $1+x-y = 2y+1$, that is, $x = 3y$. Substituting, $18y+3 = 4y+4$, so

   $$y = \frac{1}{14}, \qquad x = \frac{3}{14}$$

3. For which real numbers are the following expressions defined?

   a) $\log\log\sqrt{x}$ b) $\log\sqrt{\log x}$ c) $\sqrt{\log\log x}$

   Solution: The logarithm is defined only for positive numbers.

   a) We need $x > 0$ and $\log\sqrt{x} > 0$, that is, $\sqrt{x} > 1$: $x > 1$.

   b) We need $\sqrt{\log x} > 0$, that is, $\log x > 0$: $x > 1$.

   c) We need $\log x > 0$ and $\log\log x \geq 0$, that is, $\log x \geq 1$: $x \geq 10$.

4. In rectangle $ABCD$, $AB = 2.4 \cdot BC$. We enlarge the rectangle by a dilation with center A so that the area of the new rectangle is $\frac{9}{4}$ times the area of the original one. Because of the enlargement, the diagonal of the rectangle becomes 13 units longer. Find the sides of rectangle $ABCD$.

   Solution: The ratio of the areas of similar figures is the square of the ratio of similarity, so the ratio of the dilation is $\sqrt{\frac{9}{4}} = \frac{3}{2}$. If the original diagonal is d, then $\frac{3}{2}d - d = 13$, so $d = 26$. Let $BC = b$, so $AB = 2.4b$:

   $$d^2 = (2.4b)^2 + b^2 = 6.76b^2, \qquad d = 2.6b = 26, \qquad b = 10$$

   So $BC = 10$ and $AB = 24$ units.

5. The base ABC of the triangular pyramid $OABC$ is an isosceles right triangle with $\angle ACB = 90^\circ$. The lateral edge $OC = 3\sqrt{3}$ is perpendicular to the plane of the base. The face OAB makes an angle of 60° with the plane of the base. Find the surface area and the volume of the pyramid.

   Solution: Let $CA = CB = a$, and let M be the midpoint of AB. Then $CM \perp AB$, and by the theorem of three perpendiculars $OM \perp AB$ too, so the angle between the face OAB and the base is $\angle OMC = 60^\circ$. So

$$CM = \frac{OC}{\tan 60^\circ} = \frac{3\sqrt{3}}{\sqrt{3}} = 3, \qquad OM = \frac{OC}{\sin 60^\circ} = 6$$

   In the isosceles right triangle, $CM = \dfrac{a}{\sqrt{2}}$, so $a = 3\sqrt{2}$ and $AB = 6$. The volume is

$$V = \frac{1}{3} \cdot \frac{a^2}{2} \cdot OC = \frac{1}{3} \cdot 9 \cdot 3\sqrt{3} = 9\sqrt{3} \approx 15.59$$

   The surface consists of the base (area 9), the face OAB (area $\dfrac{1}{2} \cdot 6 \cdot 6 = 18$), and the two right triangles OCA and OCB (each with area $\dfrac{1}{2} \cdot 3\sqrt{2} \cdot 3\sqrt{3} = \dfrac{9\sqrt{6}}{2}$). So

$$A = 9 + 18 + 9\sqrt{6} = 27 + 9\sqrt{6} \approx 49.05$$

6. Let $ABCD$ be a quadrilateral in the plane. Give a construction for the point P for which $\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC} + \overrightarrow{PD} = \mathbf{0}$.

   Solution: Using position vectors, $\overrightarrow{PA} = \mathbf{a} - \mathbf{p}$, etc., so the condition is $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} - 4\mathbf{p} = \mathbf{0}$, that is,

$$\mathbf{p} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4} = \frac{1}{2}\left(\frac{\mathbf{a} + \mathbf{c}}{2} + \frac{\mathbf{b} + \mathbf{d}}{2}\right)$$

   Construction: find the midpoint E of the diagonal AC and the midpoint F of the diagonal BD; then P is the midpoint of EF. (If $E = F$, as in a parallelogram, then $P = E$.)

7. How should we choose p and q in the polynomial $x^2 + px + q$ if we want to get the polynomial $x^4 + 1$ when we multiply it by a suitably chosen quadratic polynomial?

   Solution: Since the product must have leading coefficient 1, the other factor is $x^2 + rx + s$. Comparing the coefficients of

$$\left(x^2 + px + q\right)\left(x^2 + rx + s\right) = x^4 + (p + r)\,x^3 + (q + s + pr)\,x^2 + (ps + qr)\,x + qs$$

   with $x^4 + 1$: $r = -p,\ q + s = p^2,\ p\,(s - q) = 0,\ qs = 1$.

   If $p = 0$, then $s = -q$ and $qs = -q^2 = 1$, which is impossible. So $s = q$, then $q^2 = 1$ and $2q = p^2 \geq 0$, so $q = 1$ and $p = \pm\sqrt{2}$. Indeed,

$$\left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right) = \left(x^2 + 1\right)^2 - 2x^2 = x^4 + 1$$

   So $p = \sqrt{2}$, $q = 1$ or $p = -\sqrt{2}$, $q = 1$.

8. Solve the given equation over the set of integers.

$$\cos\left(\frac{\pi}{4}\left(3x-\sqrt{9x^2-16x-80}\right)\right)=1$$

Solution: The cosine is 1 exactly when its argument is $2k\pi$, so $3x-\sqrt{9x^2-16x-80}=8k$ for some integer k, that is, $\sqrt{9x^2-16x-80}=3x-8k$, where we need $3x-8k\geq 0$. Squaring:

$$9x^2-16x-80=9x^2-48kx+64k^2,\qquad x\,(48k-16)=64k^2+80,\qquad x=\frac{4k^2+5}{3k-1}$$

Since $9\left(4k^2+5\right)=(3k-1)\,(12k+4)+49$ and $3k-1$ is not divisible by 3, x is an integer only if $3k-1$ divides 49. So $3k-1\in\{\pm1,\pm7,\pm49\}$, and k is an integer only for $3k-1=-1,-7,-49$, that is, $k=0,-2,-16$.

$k=0$: $x=-5$, but then $3x-8k=-15<0$. $k=-2$: $x=-3$, and $3x-8k=7$. $k=-16$: $x=-21$, and $3x-8k=65$.

So the solutions are $x=-3$ and $x=-21$. (Check: $\sqrt{81+48-80}=7$ and $3\,(-3)-7=-16=8\cdot(-2)$; $\sqrt{3969+336-80}=65$ and $3\,(-21)-65=-128=8\cdot(-16)$.)

6/a. A parabola with an axis parallel to the y-axis, opening upward, passes through the point $A\,(5,4)$ and is tangent to the x-axis. The tangent line of the parabola at A is perpendicular to the vector $\mathbf{v}\,(4,-1)$. Find the equation of the parabola.

Solution: Since the parabola touches the x-axis and opens upward, its equation is $y=a\,(x-h)^2$ with $a>0$. The tangent line at A has normal vector $(4,-1)$, so its slope is 4. So

$$a\,(5-h)^2=4\qquad\text{and}\qquad 2a\,(5-h)=4$$

Dividing, $\dfrac{5-h}{2}=1$, so $h=3$ and $a=1$. The parabola is $y=(x-3)^2$.

7/a. Prove that a triangle is a right triangle if and only if its acute angles α and β satisfy $\sin\alpha+\sin\beta=\cos\alpha+\cos\beta$.

Solution: By the sum-to-product identities,

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},\qquad \cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Since α and β are acute, $\left|\dfrac{\alpha-\beta}{2}\right|<45^\circ$, so $\cos\dfrac{\alpha-\beta}{2}>0$, and $0^\circ<\dfrac{\alpha+\beta}{2}<90^\circ$. So the equality holds if and only if $\tan\dfrac{\alpha+\beta}{2}=1$, that is, $\alpha+\beta=90^\circ$, that is, the third angle is 90°.

8/a. Prove that every term of the sequence

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$$

is greater than the previous term, but less than 1.

Solution: Each term is greater than the previous one:

$$a_{n+1}-a_n=\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}=\frac{1}{2n+1}-\frac{1}{2n+2}>0$$

And each term is less than 1: a_n is the sum of n fractions, each less than $\dfrac{1}{n}$, so $a_n<n\cdot\dfrac{1}{n}=1$.

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