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These problems were given in 1986 at the written mathematics entrance examination of Hungarian students applying to study at universities abroad. Problems 1–5 were the same for everyone. Of problems 6–8, the ones marked A were recommended for students who graduated from academic high schools, and the ones marked B for students who graduated from vocational secondary schools.

1. Solve the given equation over the real numbers.

$$\log\left(36^{5x}+792\right)=\log 16+\log 63$$

Solution: The right-hand side is $\log(16\cdot 63)=\log 1008$, and the logarithm is one-to-one, so

$$36^{5x}+792=1008,\qquad 36^{5x}=216,\qquad 6^{10x}=6^3,\qquad x=\frac{3}{10}$$

2. In a right triangle, the bisector of one of the acute angles divides the opposite side into two segments of length 1 and 2. Find the angles and the sides of the triangle.

Solution: Let the right angle be at C, and let the bisector of the angle α at A meet the leg BC at D. By the angle bisector theorem, $\frac{CD}{DB}=\frac{AC}{AB}$. Since the hypotenuse AB is longer than the leg AC, CD is the shorter segment: $CD=1$, $DB=2$, and $\frac{AC}{AB}=\frac{1}{2}$. So $\cos\alpha=\frac{1}{2}$, $\alpha=60^\circ$, and the angle at B is 30°.

Then $BC=3$, $AC=\frac{BC}{\tan 60^\circ}=\sqrt{3}$, and $AB=2\sqrt{3}$. The angles are 90°, 60°, 30°, and the sides are $\sqrt{3}$, 3, $2\sqrt{3}$.

3. Solve the given system of equations over the real numbers.

$$\sqrt{x+3y+1}=2,\qquad \sqrt{2x-y+2}=7y-6$$

Solution: From the first equation, $x+3y+1=4$, so $x=3-3y$. Then $2x-y+2=8-7y$, and the second equation is $\sqrt{8-7y}=7y-6$, where we need $7y-6\geq 0$. Squaring:

$$8-7y=49y^2-84y+36,\qquad 7y^2-11y+4=0,\qquad y=1 \text{ or } y=\frac{4}{7}$$

Since $7\cdot\frac{4}{7}-6<0$, only $y=1$ works, and then $x=0$. (Check: $\sqrt{4}=2$ and $\sqrt{1}=1=7-6$.)

4. Three vertices of parallelogram $ABCD$ are $A(-3,-2)$, $B(2,0)$, and $C(3,3)$. The line through B perpendicular to the diagonal AC intersects the line CD at E. Find the coordinates of E.

Solution: The fourth vertex is $D=A+C-B=(-2,1)$. The vector $\overrightarrow{AC}=(6,5)$ is a normal vector of the line through B perpendicular to AC, so this line is $6(x-2)+5y=0$, that is, $6x+5y=12$. The line CD has direction $\overrightarrow{DC}=(5,2)$, so its equation is $2x-5y=-9$. Adding the two equations, $8x=3$, so

$$E=\left(\frac{3}{8},\frac{39}{20}\right)$$

5. Of four numbers, the first three are consecutive terms of a geometric sequence, and the last three are consecutive terms of an arithmetic sequence. The sum of the first and the fourth number is 14, and the sum of the second and the third is 12. Find the four numbers.

Solution: Let the numbers be a, b, c, d. Then $c = 12 - b$, and since b, c, d is arithmetic, $d = 2c - b = 24 - 3b$. Since a, b, c is geometric, $a = \dfrac{b^2}{c} = \dfrac{b^2}{12-b}$. From $a + d = 14$:

$$\begin{aligned} \frac{b^2}{12-b} &= 3b - 10 \\ b^2 &= 46b - 3b^2 - 120 \\ 0 &= 2b^2 - 23b + 60 \\ b &= \frac{23 \pm 7}{4} \end{aligned}$$

If $b = 4$: the numbers are 2, 4, 8, 12. If $b = \dfrac{15}{2}$: the numbers are $\dfrac{25}{2}, \dfrac{15}{2}, \dfrac{9}{2}, \dfrac{3}{2}$.

6A. Solve the given system of inequalities over the real numbers.

$$x^2 - 6x - 7 < 0, \qquad \sin x > 0$$

Solution: The first inequality is $(x-7)(x+1) < 0$, that is, $-1 < x < 7$. On this interval, $\sin x > 0$ for $0 < x < \pi$ and for $2\pi < x < 7$ (since $2\pi \approx 6.28 < 7 < 3\pi$), but not for $-1 < x \leq 0$. So the solution is

$$0 < x < \pi \quad \text{or} \quad 2\pi < x < 7$$

7A. Let $ABCD$ and $EFGH$ be two parallelograms in the same plane. Prove that the midpoints of the segments AE, BF, CG, and DH, if they are not on one line, are the vertices of a parallelogram.

Solution: Use position vectors, and let K, L, M, N be the midpoints of AE, BF, CG, DH. Then $\mathbf{k} = \dfrac{\mathbf{a}+\mathbf{e}}{2}$, etc. In a parallelogram the diagonals bisect each other, so $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$ and $\mathbf{e} + \mathbf{g} = \mathbf{f} + \mathbf{h}$. So

$$\frac{\mathbf{k}+\mathbf{m}}{2} = \frac{\mathbf{a}+\mathbf{c}+\mathbf{e}+\mathbf{g}}{4} = \frac{\mathbf{b}+\mathbf{d}+\mathbf{f}+\mathbf{h}}{4} = \frac{\mathbf{l}+\mathbf{n}}{2}$$

So the segments KM and LN have the same midpoint, that is, the diagonals of the quadrilateral $KLMN$ bisect each other, and (if the points are not on one line) $KLMN$ is a parallelogram.

8A. Prove the following statement for every positive integer n: if a, b, c are integers such that n divides $a + b + c$ and $a^2 + b^2 + c^2$, then n also divides $a^4 + b^4 + c^4$.

Solution: Let $s = a + b + c$, $t = ab + bc + ca$, $u = abc$, and $p = a^2 + b^2 + c^2$. Then $p = s^2 - 2t$, so $2t = s^2 - p$ is divisible by n. Also,

$$a^2b^2 + b^2c^2 + c^2a^2 = t^2 - 2su$$

(expand t^2), and so

$$a^4 + b^4 + c^4 = p^2 - 2\left(a^2b^2 + b^2c^2 + c^2a^2\right) = p^2 - 2t^2 + 4su = p^2 - (2t)\,t + 4su$$

Each of the three terms is divisible by n (since n divides p, $2t$, and s), so n divides $a^4 + b^4 + c^4$.

6B. In the isosceles triangle ABC, the base BC is 10 units long, the altitude from A is 12, and D is the midpoint of the base. Let M be a point of the leg AC. Find the length of the segment AM if the quadrilateral $AMDB$ is cyclic.

Solution: The altitude from A goes to the midpoint D, so $AD = 12$, $DC = 5$, and $AC = 13$. The points A, M, D, B are on a circle, and C is outside it, on the lines CMA and CDB. By the theorem on secants from an external point,

$$CM \cdot CA = CD \cdot CB, \qquad CM \cdot 13 = 5 \cdot 10, \qquad CM = \frac{50}{13}$$

So $AM = 13 - \dfrac{50}{13} = \dfrac{119}{13} \approx 9.15$.

7B. Solve the given system of equations over the real numbers.

$$\frac{xy}{x+y} = 1 - z, \qquad \frac{yz}{y+z} = 2 - x, \qquad \frac{zx}{z+x} = 2 - y$$

Solution: The denominators cannot be 0. Multiplying out, the three equations are

$$xy = (1-z)(x+y), \quad yz = (2-x)(y+z), \quad zx = (2-y)(z+x)$$

Let $P = xy + yz + zx$. Rearranging, the three equations say

$$P = x + y, \qquad P = 2y + 2z, \qquad P = 2z + 2x$$

From the last two, $x = y$. Then $x + y = 2x = 2z + 2x$ gives $z = 0$, and $P = x^2 = 2x$, so $x = 0$ or $x = 2$. If $x = 0$, then $x + y = 0$, which is not allowed. So the only solution is

$$x = 2, \quad y = 2, \quad z = 0$$

(Check: $\dfrac{4}{4} = 1 - 0$, $\dfrac{0}{2} = 2 - 2$, $\dfrac{0}{2} = 2 - 2$.)

8B. Prove that if $0 \le x < \dfrac{\pi}{2}$, then

$$\frac{\sin x}{1 + \tan x} \le \frac{\sqrt{2}}{4}$$

Solution: Multiplying the numerator and the denominator by $\cos x > 0$,

$$\frac{\sin x}{1 + \tan x} = \frac{\sin x \cos x}{\sin x + \cos x}$$

Let $s = \sin x + \cos x = \sqrt{2} \sin\left(x + \dfrac{\pi}{4}\right)$; for $0 \le x < \dfrac{\pi}{2}$, $1 \le s \le \sqrt{2}$. Since $s^2 = 1 + 2 \sin x \cos x$, $\sin x \cos x = \dfrac{s^2 - 1}{2}$, and

$$\frac{\sin x \cos x}{\sin x + \cos x} = \frac{s^2 - 1}{2s} = \frac{1}{2}\left(s - \frac{1}{s}\right)$$

This is increasing in s, so it is at most $\dfrac{1}{2}\left(\sqrt{2} - \dfrac{1}{\sqrt{2}}\right) = \dfrac{\sqrt{2}}{4}$, with equality at $x = \dfrac{\pi}{4}$.

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