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These problems were given at the written mathematics entrance examination of the technical colleges in Hungary in 1986. Problems 6/a, 7/a, and 8/a were recommended for students coming from vocational secondary schools (in place of problems 6, 7, and 8).

1. Solve the given equation over the real numbers.

$$\log(4x+2) - \log(-6x) = \log(1-2x)$$

Solution: We need $4x+2>0$, $-6x>0$, and $1-2x>0$, that is, $-\frac{1}{2} < x < 0$. Then the equation is

$$\begin{aligned} \frac{4x+2}{-6x} &= 1-2x \qquad \text{multiply by } -6x \\ 4x+2 &= -6x+12x^2 \\ 0 &= 6x^2-5x-1=(6x+1)(x-1) \end{aligned}$$

Only $x=-\frac{1}{6}$ is in the domain. (Check: $\log\frac{4}{3}-\log 1=\log\frac{4}{3}$.)

2. The first term of an arithmetic sequence is -210, and its nth term is 228. The sum of the terms between them is 45. Write down the first n terms of the sequence.

Solution: The sum of the first n terms is $-210+45+228=63$, and it is also $\frac{n(-210+228)}{2}=9n$. So $n=7$, and the common difference is $d=\frac{228-(-210)}{6}=73$. The terms are

$$-210,\quad -137,\quad -64,\quad 9,\quad 82,\quad 155,\quad 228$$

3. In triangle ABC, the angle at vertex A is a right angle, and one of the angles is 30°. The distance of vertex A from the center of the inscribed circle is $2\sqrt{2}$. Find the radius of the circumscribed circle.

Solution: The center I of the inscribed circle is on the bisector of the right angle, and its distance from the legs is r. So $AI=\frac{r}{\sin 45^\circ}=r\sqrt{2}$, and $r=2$.

The angles are 90°, 60°, 30°, so if the hypotenuse is c, the legs are $\frac{c}{2}$ and $\frac{\sqrt{3}}{2}c$. In a right triangle, $r=\frac{a+b-c}{2}$, so

$$2=\frac{1}{2}\left(\frac{c}{2}+\frac{\sqrt{3}}{2}c-c\right)=\frac{\sqrt{3}-1}{4}c, \qquad c=\frac{8}{\sqrt{3}-1}=4\left(\sqrt{3}+1\right)$$

The radius of the circumscribed circle is half of the hypotenuse: $R=2\left(\sqrt{3}+1\right)\approx 5.46$.

4. For which real numbers x are the following expressions defined? What values do they take?

   a) $\sin\left(\frac{\pi}{6}\cos(\pi x)\right)$ b) $\sqrt{\sin(\pi x)}$

   Solution: a) The expression is defined for every real x. Since $\cos(\pi x)$ takes every value in $[-1,1]$, the argument $\frac{\pi}{6}\cos(\pi x)$ takes every value in $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$, where $\sin$ is increasing. So the values are $\left[-\frac{1}{2},\frac{1}{2}\right]$.

   b) We need $\sin(\pi x)\geq 0$, that is, $2k\pi\leq\pi x\leq(2k+1)\pi$, so $2k\leq x\leq 2k+1$ for some integer k. The values are $[0,1]$.

5. Two sides of a triangle are 8 and 12, and the median drawn to the third side is 9. Find the area of the triangle.

   Solution: Let c be the third side. By the formula for the median, $m_c^2=\frac{2a^2+2b^2-c^2}{4}$, so $324=128+288-c^2$ and $c^2=92$.

   A quicker way to the area: reflect the triangle about the midpoint of side c. We get a parallelogram with sides 8 and 12 and a diagonal $2\cdot 9=18$, and the triangle with sides 8, 12, 18 is half of it, just like the original triangle. By Heron's formula with $s=19$:

$$T=\sqrt{19\cdot 11\cdot 7\cdot 1}=\sqrt{1463}\approx 38.25$$

6. Solve the given inequality over the real numbers.

$$\left(x^2-x+1\right)^{x-2}>1$$

   Solution: The base $x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$ is always positive. A power of a positive base is greater than 1 exactly when either the base is greater than 1 and the exponent is positive, or the base is less than 1 and the exponent is negative.

   $x^2-x+1>1$ means $x(x-1)>0$, that is, $x<0$ or $x>1$; together with $x-2>0$: $x>2$.

   $x^2-x+1<1$ means $0<x<1$; together with $x-2<0$: $0<x<1$.

   So the solution is $0<x<1$ or $x>2$.

7. The base of a regular square pyramid is the square $ABCD$, its apex is O, and all of its edges are h units long. Find the area of the quadrilateral that the plane through the base edge AB and the midpoint F of the edge OC cuts out of the pyramid. What angle does this plane make with the base?

   Solution: The height of the pyramid is $H=\sqrt{h^2-\frac{h^2}{2}}=\frac{h}{\sqrt{2}}$. The plane intersects the face OCD in a line through F parallel to CD (since $AB\parallel CD$), so it goes through the midpoint G of OD. The cross-section is the isosceles trapezoid $ABFG$, with $AB=h$ and $GF=\frac{h}{2}$ (a midline of triangle OCD).

   The height of the trapezoid: the midpoint of GF is at height $\frac{H}{2}$ above the base, and its projection onto the base is at distance $\frac{h}{4}$ from the center of the square, on the side of CD; so it is at distance $\frac{h}{2}+\frac{h}{4}=\frac{3h}{4}$ from the line AB. So

$$m=\sqrt{\left(\frac{3h}{4}\right)^2+\left(\frac{h}{2\sqrt{2}}\right)^2}=h\sqrt{\frac{9}{16}+\frac{2}{16}}=\frac{\sqrt{11}}{4}h$$

and the area is

$$T = \frac{h + \frac{h}{2}}{2} \cdot \frac{\sqrt{11}}{4} h = \frac{3\sqrt{11}}{16} h^2 \approx 0.622 h^2$$

The angle φ between the plane and the base satisfies

$$\tan\varphi = \frac{H/2}{3h/4} = \frac{h/\left(2\sqrt{2}\right)}{3h/4} = \frac{\sqrt{2}}{3}, \qquad \varphi \approx 25.24^\circ$$

8. We write the consecutive positive integers on a board, starting with 1, up to a certain number. Then we erase one of the numbers. The arithmetic mean of the remaining numbers is $\frac{602}{17}$. Which number did we erase?

Solution: Suppose we wrote $1, 2, \ldots, n$ and erased m. The mean of the remaining $n-1$ numbers is $\frac{602}{17}$, so $n-1$ is a multiple of 17: $n - 1 = 17k$, and the sum of the remaining numbers is $602k$.

Erasing a number from $1, \ldots, n$ changes the mean, $\frac{n+1}{2}$, by less than 1 in either direction (the mean of the rest is between $\frac{n}{2}$ and $\frac{n+2}{2}$). Since $\frac{602}{17} \approx 35.4$, we get $n \approx 69$ or 70, and $n - 1 = 17k$ gives $n = 69$, $k = 4$.

Then the sum of all numbers is $\frac{69 \cdot 70}{2} = 2415$, the sum of the remaining ones is $602 \cdot 4 = 2408$, so we erased the number 7.

6/a. The height of a trapezoid inscribed in a circle is 8, and one of its legs is seen from the center of the circle at an angle of 60°. Find the area of the trapezoid.

Solution: A trapezoid inscribed in a circle is isosceles. The leg BC is seen from the center at 60°, so the inscribed angle BAC that belongs to it is 30°: the diagonal AC makes an angle of 30° with the base AB.

In an isosceles trapezoid with parallel sides a and c, the projection of a diagonal onto the longer base is $\frac{a+c}{2}$. So $\frac{a+c}{2} = \frac{8}{\tan 30^\circ} = 8\sqrt{3}$, and the area is

$$T = \frac{a+c}{2} \cdot 8 = 64\sqrt{3} \approx 110.85$$

7/a. The line g through the origin cuts a chord of length 10 from the circle $(x-5)^2 + (y-10)^2 = 50$. Find the equation of g and the coordinates of the endpoints of the chord.

Solution: The center of the circle is $(5, 10)$ and its radius is $\sqrt{50}$. A chord of length 10 is at distance $\sqrt{50 - 25} = 5$ from the center. The line $x = 0$ works: its distance from $(5, 10)$ is 5. Otherwise the line is $y = mx$, and

$$\frac{|5m - 10|}{\sqrt{m^2+1}} = 5, \qquad (5m-10)^2 = 25\left(m^2+1\right), \qquad m = \frac{3}{4}$$

So there are two such lines.

$x = 0$: $25 + (y-10)^2 = 50$, so the endpoints are $(0, 5)$ and $(0, 15)$.

$y = \frac{3}{4}x$: $(x-5)^2 + \left(\frac{3}{4}x - 10\right)^2 = 50$ simplifies to $x^2 - 16x + 48 = 0$, so $x = 4$ or $x = 12$, and the endpoints are $(4, 3)$ and $(12, 9)$.

8/a. Solve the given equation over the real numbers.

$$\sqrt{x+2+4\sqrt{x-2}}+\sqrt{x+7-6\sqrt{x-2}}=5$$

Solution: Let $t=\sqrt{x-2}\geq 0$, so $x=t^2+2$. Then

$$x+2+4\sqrt{x-2}=t^2+4t+4=(t+2)^2\,,\qquad x+7-6\sqrt{x-2}=t^2-6t+9=(t-3)^2$$

so the equation is $(t+2)+|t-3|=5$, that is, $|t-3|=3-t$. This holds exactly when $t\leq 3$. So $0\leq\sqrt{x-2}\leq 3$, and every x with $2\leq x\leq 11$ is a solution.

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