Problem Set - 1986 V.

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in subject listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Three sides of a triangle have the same lengths as the regular polygons written into a unit circle with 3 sides, 4 sides, and 6 sides, respectively. Find the lengths of the inscribed and superscribed circles.

2. Find the natural domain of the expression $\sqrt{1-\sqrt{2-\log(x-3)}}$.

3. Find the equation of the circle with the following conditions: the circle is tangent to the $x-$axis and the point of tangency is $(3,0)$. The circle has two intersections with the $y-$axis and the distance between the two are 8 units.

4. Consider the parallelogram $ABCD$. None of the parallelogram's angles is a right angle. We draw a line passing through A and perpendicular to side AB. We draw a line passing through C and perpendicular to BC. The two lines intersect each other in point E. Is line AC perpendicular to line DE?

5. Solve the given equation over the real numbers.
$$(x-1)\cdot(x-2)\cdot(x-3)\cdot(x-4)=120$$

6. The diagonals of a convex quadrilateral $ABCD$ split it into four triangles. The median points of the four triangles form another quadrilateral $PQRS$. Find the ratio between the areas of quadrilateral $ABCD$ and $PQRS$.

7. Solve the given equation over the real numbers.
$$\log_{2x}16+\log_{4x}8=\log_x 8$$

8. Consider a tetrahedron. Let m_1, m_2, m_3, and m_4 denote, for each vertex, the distance of the vertex to the face not containing that vertex. Let R denote the radius of the inscribed sphere. Prove that
$$\frac{1}{R}=\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_4}$$