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1. Two sides of an obtuse triangle are 4 and 5 units long. The sine of the angle between these sides is 0.38. Find the measures of the other two angles.

Solution: Let γ be the angle between the sides 4 and 5, and c the third side. Since $\sin \gamma = 0.38$, $\cos \gamma = \pm\sqrt{1 - 0.1444} = \pm\sqrt{0.8556} \approx \pm 0.92499$, so $\gamma \approx 22.33^\circ$ or $\gamma \approx 157.67^\circ$.

Case 1. $\gamma \approx 22.33^\circ$. By the Law of Cosines, $c^2 = 16 + 25 - 40 \cos \gamma \approx 4.000$, so $c \approx 2.000$. Then the angle opposite the side 5 satisfies $\cos \beta = \frac{16 + c^2 - 25}{2 \cdot 4 \cdot c} \approx -0.3125$, so $\beta \approx 108.21^\circ$, and the triangle is indeed obtuse. The third angle is $\alpha \approx 180^\circ - 22.33^\circ - 108.21^\circ = 49.46^\circ$.

Case 2. $\gamma \approx 157.67^\circ$. Then $c^2 = 41 + 40 \cdot 0.92499 \approx 78.00$, so $c \approx 8.832$. By the Law of Sines, the angles opposite the sides 4 and 5 satisfy $\sin \alpha = \frac{4 \cdot 0.38}{c} \approx 0.1721$ and $\sin \beta = \frac{5 \cdot 0.38}{c} \approx 0.2151$, so $\alpha \approx 9.91^\circ$ and $\beta \approx 12.42^\circ$.

So there are two triangles: the other two angles are about 49.46° and 108.21° , or about 9.91° and 12.42° .

2. Solve each of the given equations over the real numbers.

a) $x^2 + \frac{1}{x-2} = 2x - \frac{1}{2-x}$

b) $\ln(1 + \sin x) = 0.31$

Solution: a) We need $x \neq 2$. Since $-\frac{1}{2-x} = \frac{1}{x-2}$, the equation is $x^2 + \frac{1}{x-2} = 2x + \frac{1}{x-2}$, that is, $x^2 = 2x$. So $x = 0$ or $x = 2$, and since $x = 2$ is not allowed, the only solution is $x = 0$.

b) We need $1 + \sin x > 0$. Then $1 + \sin x = e^{0.31} \approx 1.3634$, so $\sin x \approx 0.3634$, and

$$x \approx 0.3719 + 2k\pi \quad \text{or} \quad x \approx \pi - 0.3719 + 2k\pi \approx 2.7697 + 2k\pi$$

where k is any integer (in degrees: $x \approx 21.31^\circ + k \cdot 360^\circ$ or $x \approx 158.69^\circ + k \cdot 360^\circ$).

3. In an arithmetic sequence $a_p = q$ and $a_q = p$. Express a_n in terms of p , q , and n .

Solution: We assume $p \neq q$. The common difference is

$$d = \frac{a_q - a_p}{q - p} = \frac{p - q}{q - p} = -1$$

So $a_n = a_p + (n - p)d = q - (n - p) = p + q - n$.

4. One leg of a right triangle is the geometric mean of the hypotenuse and the other leg. Find the ratio of the radii of the inscribed and circumscribed circles.

Solution: Let the legs be a and b , and the hypotenuse c , with $a^2 = bc$. By the Pythagorean theorem,

$$c^2 = a^2 + b^2 = bc + b^2$$

Dividing by c^2 and letting $t = \frac{b}{c}$: $t^2 + t - 1 = 0$, so $t = \frac{\sqrt{5}-1}{2} \approx 0.618$. Then $\frac{a}{c} = \sqrt{\frac{bc}{c^2}} = \sqrt{t}$.

In a right triangle, $R = \frac{c}{2}$ and $r = \frac{a+b-c}{2}$, so

$$\frac{r}{R} = \frac{a+b-c}{c} = \sqrt{t} + t - 1 = \sqrt{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2} \approx 0.404$$

5. Solve the given equation over the real numbers.

$$\sin^6 x + \cos^6 x = \frac{7}{16}$$

Solution: Using $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = \sin^2 x$, $b = \cos^2 x$, and $a+b=1$:

$$\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x = 1 - \frac{3}{4}\sin^2 2x$$

So the equation is

$$1 - \frac{3}{4}\sin^2 2x = \frac{7}{16}, \quad \sin^2 2x = \frac{3}{4}, \quad \sin 2x = \pm \frac{\sqrt{3}}{2}$$

So $2x = \pm \frac{\pi}{3} + k\pi$, and

$$x = \pm \frac{\pi}{6} + k \cdot \frac{\pi}{2} \quad \text{where } k \text{ is any integer.}$$

6. Consider a circle with radius R . Let P be a point on the circle. We draw chords PA , PB , PC so that $\angle APB = \angle BPC = 60^\circ$. Find the area of triangle ABC .

Solution: (We assume that PB is between PA and PC , so $\angle APC = 120^\circ$.) By the Law of Sines in a circle, a chord that subtends an inscribed angle φ has length $2R \sin \varphi$. So

$$AB = 2R \sin 60^\circ = R\sqrt{3}, \quad BC = 2R \sin 60^\circ = R\sqrt{3}, \quad AC = 2R \sin 120^\circ = R\sqrt{3}$$

So triangle ABC is equilateral with side $R\sqrt{3}$, and its area is

$$T = \frac{\sqrt{3}}{4} (R\sqrt{3})^2 = \frac{3\sqrt{3}}{4} R^2$$

7. In triangle ABC , let E_1 denote the point where the inscribed circle is tangent to side BC . Let E_2 denote the point where side BC is tangent to the circle that is tangent to side BC from outside and to the extensions of the other two sides. (This second circle is called the excircle drawn to side BC .) Express the distance E_1E_2 in terms of the sides of the triangle.

Solution: Let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{a+b+c}{2}$. The two tangent segments from a point to a circle are equal.

Inscribed circle: let the tangent segments from A, B, C be x, y, z . Then $x+y = c$, $y+z = a$, $z+x = b$, so $y = s - b$. So $BE_1 = s - b$.

Excircle drawn to side BC : it is tangent to the extensions of AB and AC beyond B and C , at points F and G . Then $AF = AG$, and since $BF = BE_2$ and $CG = CE_2$,

$$AF + AG = (c + BE_2) + (b + CE_2) = a + b + c = 2s$$

so $AF = s$ and $BE_2 = BF = s - c$.

Both E_1 and E_2 are on side BC , so

$$E_1E_2 = |BE_1 - BE_2| = |(s - b) - (s - c)| = |b - c|$$

8. Consider the parabola with equation $y = ax^2 + bx + c$. This parabola passes through the points $(0, k)$, $(k, 2k)$, and $(2k, 0)$. Find all values of k for which the parabola intersects the x -axis in $(-1, 0)$. In this case, what are the values of a , b , and c ?

Solution: We need $k \neq 0$ (otherwise the three points are the same). From the point $(0, k)$, $c = k$. From the other two points,

$$\begin{aligned} ak^2 + bk + k &= 2k & \text{so} & \quad ak + b = 1 \\ 4ak^2 + 2bk + k &= 0 & \text{so} & \quad 4ak + 2b + 1 = 0 \end{aligned}$$

Substituting $b = 1 - ak$ into the second equation: $2ak + 3 = 0$, so $ak = -\frac{3}{2}$, $a = -\frac{3}{2k}$, and $b = \frac{5}{2}$. The parabola passes through $(-1, 0)$ if $a - b + c = 0$:

$$\begin{aligned} -\frac{3}{2k} - \frac{5}{2} + k &= 0 & \text{multiply by } 2k \\ 2k^2 - 5k - 3 &= 0 \\ (2k + 1)(k - 3) &= 0 \end{aligned}$$

If $k = 3$: $a = -\frac{1}{2}$, $b = \frac{5}{2}$, $c = 3$, so $y = -\frac{1}{2}x^2 + \frac{5}{2}x + 3$.

If $k = -\frac{1}{2}$: $a = 3$, $b = \frac{5}{2}$, $c = -\frac{1}{2}$, so $y = 3x^2 + \frac{5}{2}x - \frac{1}{2}$. (In this case the point $(2k, 0)$ is $(-1, 0)$ itself.)