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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in subject listed above) add up to a total of 60 points. In addition, student assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The three sides of a triangle have the same lengths as the sides of the regular polygons inscribed in a unit circle with 3 sides, 4 sides, and 6 sides, respectively. Find the radii of the inscribed and circumscribed circles of the triangle.

Solution: The side of a regular n -gon inscribed in a unit circle is $2 \sin \frac{180^\circ}{n}$. So the sides of the triangle are

$$2 \sin 60^\circ = \sqrt{3}, \quad 2 \sin 45^\circ = \sqrt{2}, \quad \text{and} \quad 2 \sin 30^\circ = 1$$

Since $1^2 + (\sqrt{2})^2 = 3 = (\sqrt{3})^2$, the triangle is a right triangle with hypotenuse $\sqrt{3}$. The center of the circumscribed circle is the midpoint of the hypotenuse, so its radius is $R = \frac{\sqrt{3}}{2} \approx 0.866$. In a right triangle, the radius of the inscribed circle is $r = \frac{a + b - c}{2}$ (the two tangent segments from the right angle vertex are both r), so

$$r = \frac{1 + \sqrt{2} - \sqrt{3}}{2} \approx 0.341$$

2. Find the natural domain of the expression $\sqrt{1 - \sqrt{2 - \log(x - 3)}}$.

Solution: Here $\log$ is the logarithm with base 10. We need three conditions.

- $x - 3 > 0$, that is, $x > 3$.
- $2 - \log(x - 3) \geq 0$, that is, $\log(x - 3) \leq 2$, so $x - 3 \leq 100$ and $x \leq 103$.
- $1 - \sqrt{2 - \log(x - 3)} \geq 0$, that is, $2 - \log(x - 3) \leq 1$, so $\log(x - 3) \geq 1$, $x - 3 \geq 10$, and $x \geq 13$.

So the domain is $13 \leq x \leq 103$.

3. Find the equation of the circle with the following conditions: the circle is tangent to the x -axis and the point of tangency is $(3, 0)$. The circle intersects the y -axis in two points, and the distance between these two points is 8 units.

Solution: The center is on the line $x = 3$ (perpendicular to the x -axis at the point of tangency), so the center is $(3, k)$ and the radius is $r = |k|$. The distance of the center from the y -axis is 3, and the chord on the y -axis is 8 units long, so half of the chord is 4. By the Pythagorean theorem, $r^2 = 3^2 + 4^2 = 25$, so $r = 5$ and $k = \pm 5$. There are two such circles:

$$(x - 3)^2 + (y - 5)^2 = 25 \quad \text{and} \quad (x - 3)^2 + (y + 5)^2 = 25$$

4. Consider the parallelogram $ABCD$. None of the parallelogram's angles is a right angle. We draw a line passing through A and perpendicular to side AB . We draw a line passing through C and perpendicular to BC . The two lines intersect each other in point E . Is line AC perpendicular to line DE ?

Solution: Yes, AC and DE are always perpendicular. We use vectors with the origin at B : let $\mathbf{a} = \overrightarrow{BA}$ and $\mathbf{c} = \overrightarrow{BC}$. Then $\overrightarrow{BD} = \mathbf{a} + \mathbf{c}$, and let $\mathbf{e} = \overrightarrow{BE}$. (The two perpendicular lines do intersect, because AB and BC are not parallel.) Since $EA \perp AB$: $(\mathbf{e} - \mathbf{a}) \cdot \mathbf{a} = 0$, so $\mathbf{e} \cdot \mathbf{a} = |\mathbf{a}|^2$. Since $EC \perp BC$: $(\mathbf{e} - \mathbf{c}) \cdot \mathbf{c} = 0$, so $\mathbf{e} \cdot \mathbf{c} = |\mathbf{c}|^2$. Then

$$\begin{aligned}\overrightarrow{DE} \cdot \overrightarrow{AC} &= (\mathbf{e} - \mathbf{a} - \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = \mathbf{e} \cdot \mathbf{c} - \mathbf{e} \cdot \mathbf{a} - (\mathbf{c} + \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) \\ &= |\mathbf{c}|^2 - |\mathbf{a}|^2 - (|\mathbf{c}|^2 - |\mathbf{a}|^2) = 0\end{aligned}$$

So $DE \perp AC$.

5. Solve the given equation over the real numbers.

$$(x-1) \cdot (x-2) \cdot (x-3) \cdot (x-4) = 120$$

Solution: We multiply the first and last factors, and the two middle ones:

$$(x-1)(x-4) = x^2 - 5x + 4, \quad (x-2)(x-3) = x^2 - 5x + 6$$

Let $t = x^2 - 5x + 5$. Then the equation is

$$(t-1)(t+1) = 120, \quad t^2 = 121, \quad t = \pm 11$$

If $t = 11$: $x^2 - 5x - 6 = 0$, that is, $(x-6)(x+1) = 0$, so $x = 6$ or $x = -1$. If $t = -11$: $x^2 - 5x + 16 = 0$ has no real solution (its discriminant is $25 - 64 < 0$). So the solutions are $x = 6$ and $x = -1$.

6. The diagonals of a convex quadrilateral $ABCD$ split it into four triangles. The centroids of the four triangles form another quadrilateral $PQRS$. Find the ratio between the areas of quadrilateral $ABCD$ and $PQRS$.

Solution: Let O be the intersection of the diagonals, and use vectors with the origin at O . The centroid of a triangle is the average of its vertices, so the centroids of triangles ABO , BCO , CDO , and DAO are

$$P = \frac{A+B}{3}, \quad Q = \frac{B+C}{3}, \quad R = \frac{C+D}{3}, \quad S = \frac{D+A}{3}$$

Then $\overrightarrow{PQ} = \overrightarrow{SR} = \frac{1}{3}\overrightarrow{AC}$ and $\overrightarrow{QR} = \overrightarrow{PS} = \frac{1}{3}\overrightarrow{BD}$. So $PQRS$ is a parallelogram whose sides are parallel to the diagonals of $ABCD$ and are one third of them. If φ is the angle between the diagonals, then

$$T_{ABCD} = \frac{1}{2} \cdot AC \cdot BD \cdot \sin \varphi, \quad T_{PQRS} = \frac{AC}{3} \cdot \frac{BD}{3} \cdot \sin \varphi$$

and so the ratio of the areas is $\frac{T_{ABCD}}{T_{PQRS}} = \frac{1/2}{1/9} = \frac{9}{2}$, that is, $9 : 2$.

7. Solve the given equation over the real numbers.

$$\log_{2x} 16 + \log_{4x} 8 = \log_x 8$$

Solution: We need $x > 0$, and the bases cannot be 1: $x \neq 1, \frac{1}{2}, \frac{1}{4}$. Let $t = \log_2 x$; then $t \neq 0, -1, -2$. Changing to base 2,

$$\log_{2x} 16 = \frac{4}{1+t}, \quad \log_{4x} 8 = \frac{3}{2+t}, \quad \log_x 8 = \frac{3}{t}$$

$$\begin{aligned} \frac{4}{1+t} + \frac{3}{2+t} &= \frac{3}{t} && \text{multiply by } t(1+t)(2+t) \\ 4t(2+t) + 3t(1+t) &= 3(1+t)(2+t) \\ 7t^2 + 11t &= 3t^2 + 9t + 6 \\ 2t^2 + t - 3 &= 0 \\ (2t+3)(t-1) &= 0 \end{aligned}$$

So $t = 1$ or $t = -\frac{3}{2}$, that is, $x = 2$ or $x = 2^{-\frac{3}{2}} = \frac{\sqrt{2}}{4}$. (Check with $x = 2$: $\log_4 16 + \log_8 8 = 2 + 1 = 3 = \log_2 8$.)

8. Consider a tetrahedron. Let m_1, m_2, m_3 , and m_4 denote, for each vertex, the distance of the vertex to the face not containing that vertex. Let R denote the radius of the inscribed sphere. Prove that

$$\frac{1}{R} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}$$

Solution: Let A_1, A_2, A_3, A_4 be the areas of the faces, where A_i is the face opposite the vertex with height m_i , and let V be the volume. Then $V = \frac{1}{3}A_i m_i$ for each i , so $A_i = \frac{3V}{m_i}$.

Let O be the center of the inscribed sphere. Connecting O with the four vertices splits the tetrahedron into four pyramids, whose bases are the faces and whose heights are all R . So

$$V = \frac{1}{3}R(A_1 + A_2 + A_3 + A_4) = \frac{1}{3}R\left(\frac{3V}{m_1} + \frac{3V}{m_2} + \frac{3V}{m_3} + \frac{3V}{m_4}\right) = RV\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}\right)$$

Dividing by RV gives $\frac{1}{R} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4}$.