

author: Imre Rábai

translated from: KoMaL, December 1987, pages 439-440

Pdf file: KoMaL, December 1987, pages 439-440

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given inequality over the real numbers.

$$(x-1)^2 + 2 \leq 2\sqrt{x^2}$$

Solution: $\sqrt{x^2} = |x|$.

If $x \geq 0$: $x^2 - 2x + 3 \leq 2x$, that is, $x^2 - 4x + 3 = (x-1)(x-3) \leq 0$, so $1 \leq x \leq 3$.

If $x < 0$: $x^2 - 2x + 3 \leq -2x$, that is, $x^2 + 3 \leq 0$, which is never true.

The solution is $1 \leq x \leq 3$.

2. Find the equation of the circle that passes through the point $A(2, 1)$, is tangent to the x -axis, and whose center is on the line $x - 2y = 1$.

Solution: Let the center be (h, k) . Since the circle is tangent to the x -axis, its radius is $|k|$. The center is on the line, so $h = 1 + 2k$. The point A is on the circle:

$$(2-h)^2 + (1-k)^2 = k^2, \quad (1-2k)^2 + (1-k)^2 = k^2, \quad 2k^2 - 3k + 1 = 0$$

so $k = 1$ or $k = \frac{1}{2}$. There are two such circles:

$$(x-3)^2 + (y-1)^2 = 1, \quad (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

3. Find the exact value of $\sin x$ if $\tan 2x + 2 \tan x = 0$.

Solution: Let $t = \tan x$ (we need $t \neq \pm 1$ so that $\tan 2x$ is defined). Then

$$\frac{2t}{1-t^2} + 2t = 0, \quad 2t \cdot \frac{2-t^2}{1-t^2} = 0$$

so $t = 0$ or $t^2 = 2$. If $\tan x = 0$, then $\sin x = 0$. If $\tan^2 x = 2$, then

$$\sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x} = \frac{2}{3}, \quad \sin x = \pm \frac{\sqrt{6}}{3}$$

(both signs occur, for example for $x = \tan^{-1} \sqrt{2}$ and $x = -\tan^{-1} \sqrt{2}$). So $\sin x$ is 0, $\frac{\sqrt{6}}{3}$, or $-\frac{\sqrt{6}}{3}$.

4. Solve the given equation over the real numbers.

$$x^{2-\log \frac{x}{2}} = 4$$

Solution: We need $x > 0$. Taking the (base 10) logarithm of both sides and using $\log \frac{x}{2} = \log x - \log 2$:

$$(2 - \log x + \log 2) \log x = 2 \log 2$$

With $u = \log x$ and $c = \log 2$:

$$u^2 - (2 + c)u + 2c = 0, \quad (u - 2)(u - c) = 0$$

So $\log x = 2$ or $\log x = \log 2$, that is, $x = 100$ or $x = 2$. Check: $2^{2-\log 1} = 2^2 = 4$, and $100^{2-\log 50} = 100^{\log 2} = 10^{2\log 2} = 4$.

5. In triangle ABC , $BC = \sqrt{10}$ units, the radius of the circumscribed circle is $r = \sqrt{5}$ units, and $AB = 2\sqrt{2} \cdot AC$. Find the length of the side AB .

Solution: By the Law of Sines, $\sin \alpha = \frac{BC}{2r} = \frac{\sqrt{10}}{2\sqrt{5}} = \frac{\sqrt{2}}{2}$, so the angle at A is $\alpha = 45^\circ$ or $\alpha = 135^\circ$. Let $AC = b$, so $AB = 2\sqrt{2}b$. By the Law of Cosines,

$$10 = b^2 + 8b^2 - 2 \cdot b \cdot 2\sqrt{2}b \cos \alpha = 9b^2 - 4\sqrt{2}b^2 \cos \alpha$$

If $\alpha = 45^\circ$: $10 = 9b^2 - 4b^2 = 5b^2$, so $b = \sqrt{2}$ and $AB = 4$.

If $\alpha = 135^\circ$: $10 = 9b^2 + 4b^2 = 13b^2$, so $b = \sqrt{\frac{10}{13}}$ and $AB = 2\sqrt{2}\sqrt{\frac{10}{13}} = \frac{4\sqrt{65}}{13} \approx 2.48$.

Both triangles exist, so $AB = 4$ or $AB = \frac{4\sqrt{65}}{13}$.

6. In rectangle $ABCD$, $AB = 2 \cdot BC$. The distances of the vertices C and D from a point P of the line AB are 5 and $\sqrt{41}$ units. Find the lengths of the sides of the rectangle.

Solution: Let $BC = s$ and use coordinates: $A(0, 0)$, $B(2s, 0)$, $C(2s, s)$, $D(0, s)$, and $P(p, 0)$. Then

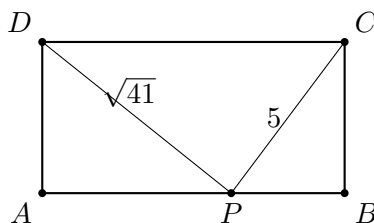
$$PC^2 = (2s - p)^2 + s^2 = 25, \quad PD^2 = p^2 + s^2 = 41$$

Subtracting the second equation from the first one: $4s^2 - 4sp = -16$, so $p = \frac{s^2 + 4}{s}$. Substituting this into the second equation and multiplying by s^2 :

$$(s^2 + 4)^2 + s^4 = 41s^2, \quad 2s^4 - 33s^2 + 16 = 0, \quad s^2 = \frac{33 \pm 31}{4}$$

If $s^2 = 16$: the sides are $BC = 4$ and $AB = 8$ (with $P(5, 0)$).

If $s^2 = \frac{1}{2}$: the sides are $BC = \frac{\sqrt{2}}{2}$ and $AB = \sqrt{2}$ (with $p = \frac{9\sqrt{2}}{2}$, P is on the extension of AB beyond B).



7. Solve the given system of equations, where a is a real parameter.

$$\begin{aligned}x^2 + y^2 &= 16 \\(x - 2a)(ay + x) &= 0\end{aligned}$$

Solution: The second equation says $x = 2a$ or $x = -ay$.

If $x = 2a$: $y^2 = 16 - 4a^2$, which has solutions only if $|a| \leq 2$: $(2a, 2\sqrt{4 - a^2})$ and $(2a, -2\sqrt{4 - a^2})$.

If $x = -ay$: $(a^2 + 1)y^2 = 16$, so for every a we get the two solutions

$$\left(-\frac{4a}{\sqrt{a^2 + 1}}, \frac{4}{\sqrt{a^2 + 1}}\right), \quad \left(\frac{4a}{\sqrt{a^2 + 1}}, -\frac{4}{\sqrt{a^2 + 1}}\right)$$

Some of these solutions can coincide. A solution of the first kind is also of the second kind if $2a = -ay$: for $a = 0$, the two pairs are the same, $(0, \pm 4)$; for $a \neq 0$ we need $y = -2$, so $16 - 4a^2 = 4$, $a = \pm\sqrt{3}$. Also, for $a = \pm 2$ the two solutions of the first kind coincide: $(\pm 4, 0)$. So the number of solutions is

- 2 if $|a| > 2$ or $a = 0$,
- 3 if $a = \pm 2$ or $a = \pm\sqrt{3}$,
- 4 for every other a with $|a| < 2$.

8. The real numbers x, y , and z satisfy the following equations.

$$\begin{aligned}x + y &= z - 1 \\xy &= z^2 - 7z + 14\end{aligned}$$

For what value of z is the sum $x^2 + y^2$ the largest? What is the largest value of $x^2 + y^2$?

Solution: We have

$$x^2 + y^2 = (x + y)^2 - 2xy = (z - 1)^2 - 2(z^2 - 7z + 14) = -z^2 + 12z - 27 = 9 - (z - 6)^2$$

This would be largest at $z = 6$, but we must be careful: x and y are the roots of the quadratic equation $t^2 - (z - 1)t + (z^2 - 7z + 14) = 0$, and they are real only if its discriminant is not negative:

$$(z - 1)^2 - 4(z^2 - 7z + 14) \geq 0, \quad 3z^2 - 26z + 55 \leq 0, \quad \frac{11}{3} \leq z \leq 5$$

(For $z = 6$, the discriminant is -15 , and there are no real x and y .) On the interval $\left[\frac{11}{3}, 5\right]$, the function $9 - (z - 6)^2$ is increasing, so the largest value is at $z = 5$:

$$x^2 + y^2 = 9 - 1 = 8$$

In this case $x + y = 4$ and $xy = 4$, so $x = y = 2$.