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translated from: KoMaL, January 1987, pages 17-18

Pdf file: KoMaL, January 1987, pages 17-18

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation.

$$x^2 - 5x + 6 = 1 - \frac{1}{x^2 - 5x + 7}$$

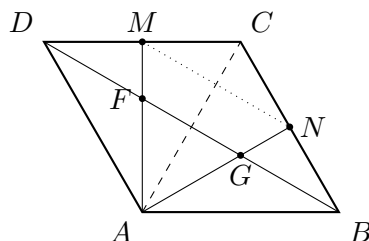
Solution: Let $u = x^2 - 5x + 7$. Since the discriminant $25 - 28$ is negative, u is always positive, so the denominator is never zero. The equation is

$$\begin{aligned} u - 1 &= 1 - \frac{1}{u} \quad \text{multiply by } u \\ u^2 - 2u + 1 &= 0 \\ (u - 1)^2 &= 0 \end{aligned}$$

So $u = 1$, that is, $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$, and the solutions are $x = 2$ and $x = 3$.

2. In rhombus $ABCD$, $AC = AB$. We draw the perpendiculars from vertex A to the sides DC and BC , with feet M and N . Let F and G be the points where the lines AM and AN intersect the diagonal BD . Prove that $FG : MN = 2 : 3$.

Solution: Since $AC = AB = BC = CD = DA$, the triangles ABC and ACD are equilateral. In an equilateral triangle the altitude is also a median, so M is the midpoint of DC and N is the midpoint of BC .



So MN is a midline of triangle BCD , and $MN = \frac{BD}{2}$.

Let O be the intersection of the diagonals; it is the midpoint of AC . In triangle ACD , both AM and DO are medians, so their intersection F is the centroid of ACD , and $DF = \frac{2}{3}DO = \frac{1}{3}BD$. The same way, G is the centroid of triangle ABC , and $BG = \frac{1}{3}BD$. So

$$FG = BD - DF - BG = \frac{1}{3}BD, \quad \text{and} \quad FG : MN = \frac{1}{3} : \frac{1}{2} = 2 : 3$$

3. Without using approximate values, decide which number is greater:

$$A = \left(\sqrt{2} - \log_2 \sin \frac{29\pi}{6} \right)^2 \quad \text{or} \quad B = \frac{\sqrt{8}}{2 - \sqrt{2}}$$

Solution: (In the original, the angle is printed as $\frac{29\pi}{5}$. Then $\sin \frac{29\pi}{5} = \sin \left(6\pi - \frac{\pi}{5} \right) = -\sin \frac{\pi}{5} < 0$, and A would not be defined, so this is evidently a misprint for $\frac{29\pi}{6}$.)

Since $\frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$, $\sin \frac{29\pi}{6} = \sin \frac{5\pi}{6} = \frac{1}{2}$, and $\log_2 \frac{1}{2} = -1$. So

$$A = \left(\sqrt{2} + 1 \right)^2 = 3 + 2\sqrt{2}$$

For B , we rationalize the denominator:

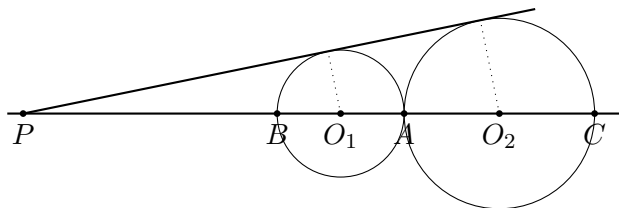
$$B = \frac{2\sqrt{2}}{2 - \sqrt{2}} \cdot \frac{2 + \sqrt{2}}{2 + \sqrt{2}} = \frac{2\sqrt{2}(2 + \sqrt{2})}{2} = 2\sqrt{2} + 2$$

So $A - B = 1$, and A is greater.

4. Two circles with different radii are tangent to each other from outside at point A . The line through their centers intersects the smaller circle again at B , the larger circle again at C , and one of their common external tangent lines at P . Find the radii of the circles if $PB = 8$ and $PC = 18$.

Solution: Let $r < R$ be the radii, and O_1, O_2 the centers of the smaller and larger circles. The point P is on the side of the smaller circle, so the order of the points on the line is P, B, O_1, A, O_2, C , and

$$BC = 2r + 2R = 18 - 8 = 10, \quad \text{so} \quad r + R = 5$$



The radii drawn to the points of tangency on the common tangent are perpendicular to it, so they are parallel, and the right triangles they form with P are similar. So $\frac{PO_1}{r} = \frac{PO_2}{R}$, where $PO_1 = 8 + r$ and $PO_2 = 8 + 2r + R$. With $R = 5 - r$:

$$\begin{aligned} \frac{8+r}{r} &= \frac{13+r}{5-r} \\ (8+r)(5-r) &= r(13+r) \\ 40 - 3r - r^2 &= 13r + r^2 \\ 0 &= r^2 + 8r - 20 = (r-2)(r+10) \end{aligned}$$

So $r = 2$ and $R = 3$. (Check: $PO_1 = 10 = 5r$ and $PO_2 = 15 = 5R$.)

5. For which real numbers x is it true that

$$\log_3 (9^{2x} - 3^{2x+1} + 3) < 2 \log_9 7 ?$$

Solution: Since $2 \log_9 7 = \log_3 7$ and the base 3 is greater than 1, the inequality is equivalent to $0 < 9^{2x} - 3^{2x+1} + 3 < 7$. Let $u = 3^{2x} > 0$. Then $9^{2x} = u^2$ and $3^{2x+1} = 3u$, so we need

$$0 < u^2 - 3u + 3 < 7$$

The left inequality always holds (the discriminant $9 - 12$ is negative). The right one is $u^2 - 3u - 4 = (u - 4)(u + 1) < 0$, so $u < 4$ (since $u > 0$). So $3^{2x} < 4$, $2x < \log_3 4$, and

$$x < \log_3 2$$

6. Prove that for any point inside a regular tetrahedron with edges $\frac{\sqrt{6}}{2}$ units long, the sum of the reciprocals of its distances from the faces is at least 16.

Solution: First we find the height of the tetrahedron. The center of a face (an equilateral triangle with side a) is at distance $\frac{a}{\sqrt{3}}$ from its vertices, so the height is

$$m = \sqrt{a^2 - \frac{a^2}{3}} = a\sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{2} \cdot \frac{\sqrt{2}}{\sqrt{3}} = 1$$

Let P be a point inside, and d_1, d_2, d_3, d_4 its distances from the faces. Connecting P with the vertices splits the tetrahedron into four pyramids with the faces as bases. All faces have the same area T , so

$$\frac{1}{3}Tm = V = \frac{1}{3}T(d_1 + d_2 + d_3 + d_4), \quad \text{so} \quad d_1 + d_2 + d_3 + d_4 = m = 1$$

By the inequality between the arithmetic and harmonic means (or by the Cauchy-Schwarz inequality),

$$(d_1 + d_2 + d_3 + d_4) \left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_4} \right) \geq 16$$

and since the first factor is 1, the sum of the reciprocals is at least 16. (Equality holds at the center, where all distances are $\frac{1}{4}$.)

7. Let S_n be the sum of the first n terms of a geometric sequence, and Q_n the sum of the reciprocals of the first n terms. Find the product of the first n terms.

Solution: Let the first term be a and the common ratio q . If $q \neq 1$,

$$S_n = a \cdot \frac{q^n - 1}{q - 1}, \quad Q_n = \frac{1}{a} \cdot \frac{q^{-n} - 1}{q^{-1} - 1} = \frac{1}{a} \cdot \frac{q^n - 1}{(q - 1)q^{n-1}}$$

so $\frac{S_n}{Q_n} = a^2 q^{n-1}$; this is also true if $q = 1$ (then $S_n = na$ and $Q_n = \frac{n}{a}$). Notice that $a^2 q^{n-1} = a \cdot a q^{n-1}$ is the product of the first and the n th term. The product of the first n terms is

$$P_n = a \cdot a q \cdot a q^2 \cdots a q^{n-1} = a^n q^{\frac{n(n-1)}{2}}, \quad \text{so} \quad P_n^2 = (a^2 q^{n-1})^n = \left(\frac{S_n}{Q_n} \right)^n$$

If the terms are positive, then

$$P_n = \left(\frac{S_n}{Q_n} \right)^{\frac{n}{2}}$$

(In general, $|P_n| = \left(\frac{S_n}{Q_n} \right)^{\frac{n}{2}}$, and the sign of P_n is the sign of the product of the terms.)

8. How many four-digit numbers between 1000 and 9999 have all different digits and the absolute value of the difference of the first and last digits equal to 2?

Solution: First we count the possible pairs (first digit, last digit). The first digit is 1 to 9, the last digit is 0 to 9, and they differ by 2:

- first digit 1: last digit 3 (1 pair);
- first digit 2: last digit 0 or 4 (2 pairs);
- first digit 3, 4, 5, 6, or 7: last digit two less or two more (10 pairs);
- first digit 8: last digit 6 (1 pair);
- first digit 9: last digit 7 (1 pair).

That is 15 pairs. For each pair, the two middle digits must be different from each other and from the first and last digits, so there are $8 \cdot 7 = 56$ ways to choose them. So the number of such four-digit numbers is $15 \cdot 56 = 840$.