

author: László Bánhegyi

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation.

$$\log_{x-4} (x-4)^7 - 2x - \frac{1-3x}{7} = 2 - \frac{2x-1}{3}$$

Solution: The base of the logarithm must be positive and not 1, so $x > 4$ and $x \neq 5$. Then $\log_{x-4} (x-4)^7 = 7$, and the equation is

$$\begin{aligned} 7 - 2x - \frac{1-3x}{7} &= 2 - \frac{2x-1}{3} && \text{multiply by 21} \\ 147 - 42x - 3 + 9x &= 42 - 14x + 7 \\ 95 &= 19x \\ x &= 5 \end{aligned}$$

But $x = 5$ is not in the domain (the base would be 1). So the equation has no solution.

2. The parallel sides of a trapezoid are 4 and 7 units long, and its legs are 4 and 5 units long. How far is the intersection of the diagonals from the sides of the trapezoid?

Solution: Let the parallel sides be $AB = 7$ and $CD = 4$, the legs $AD = 4$ and $BC = 5$, and the height h . The projections of the legs onto AB add up to $7 - 4 = 3$. If they are x and y , then $x^2 + h^2 = 16$ and $y^2 + h^2 = 25$, so $y^2 - x^2 = (y-x)(y+x) = 9$, and since $x+y = 3$, $y-x = 3$. So $x = 0$ and $y = 3$: the trapezoid is a right trapezoid with height $h = 4$.

We use coordinates: $A(0,0)$, $B(7,0)$, $C(4,4)$, $D(0,4)$. The diagonal AC is $y = x$, and BD is $\frac{x}{7} + \frac{y}{4} = 1$. Their intersection is $M\left(\frac{28}{11}, \frac{28}{11}\right)$. The distances of M :

- from AB ($y = 0$): $\frac{28}{11} \approx 2.55$;
- from CD ($y = 4$): $4 - \frac{28}{11} = \frac{16}{11} \approx 1.45$;
- from AD ($x = 0$): $\frac{28}{11} \approx 2.55$;
- from BC ($4x + 3y = 28$): $\frac{\left|4 \cdot \frac{28}{11} + 3 \cdot \frac{28}{11} - 28\right|}{5} = \frac{112}{55} \approx 2.04$.

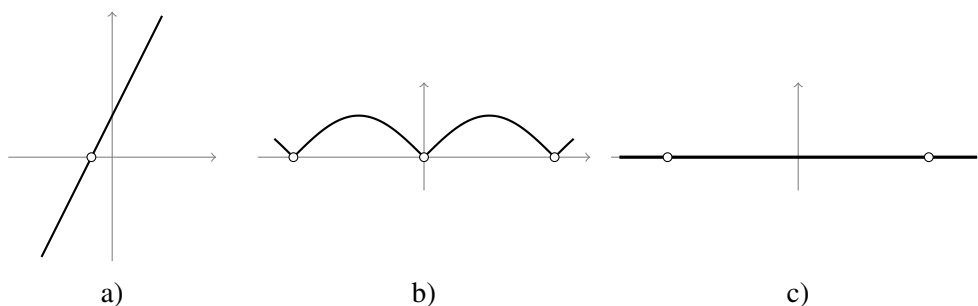
3. Graph each of the following.

a) the function f for which $1 = \frac{2x+1}{f(x)}$ b) $x \mapsto 10^{\frac{1}{2} \log \sin^2 x}$ c) $x \mapsto \log_{1986} \sqrt{\frac{1+\cos x}{2 \cos^2 \frac{x}{2}}}$

Solution: a) From $1 = \frac{2x+1}{f(x)}$, $f(x) = 2x+1$, but $f(x)$ cannot be 0, so $x \neq -\frac{1}{2}$. The graph is the line $y = 2x+1$ with the point $(-\frac{1}{2}, 0)$ removed.

b) The logarithm is defined if $\sin^2 x > 0$, that is, $x \neq k\pi$. Then $10^{\frac{1}{2} \log \sin^2 x} = (\sin^2 x)^{\frac{1}{2}} = |\sin x|$. The graph is the graph of $y = |\sin x|$ with the points $(k\pi, 0)$ removed.

c) The function is defined if $\cos \frac{x}{2} \neq 0$, that is, $x \neq \pi + 2k\pi$. Since $1 + \cos x = 2 \cos^2 \frac{x}{2}$, the fraction is 1, and the function value is $\log_{1986} 1 = 0$. The graph is the x -axis with the points $(\pi + 2k\pi, 0)$ removed.



4. In what base is the number 1331 a perfect cube?

Solution: In base b , the digit 3 appears, so $b \geq 4$. Then

$$(1331)_b = b^3 + 3b^2 + 3b + 1 = (b+1)^3$$

So 1331 is a perfect cube in every base $b \geq 4$: it is the cube of $(11)_b = b+1$.

5. Prove that $x + y + xy \leq x^2 + y^2 + 1$ for all real numbers x and y .

Solution: We show that twice the difference of the two sides is a sum of squares:

$$\begin{aligned} 2(x^2 + y^2 + 1 - x - y - xy) &= (x^2 - 2xy + y^2) + (x^2 - 2x + 1) + (y^2 - 2y + 1) \\ &= (x - y)^2 + (x - 1)^2 + (y - 1)^2 \geq 0 \end{aligned}$$

So $x + y + xy \leq x^2 + y^2 + 1$, with equality only if $x = y = 1$.

6. The sides of rectangle $ABCD$ are $AB = CD = 38$ and $BC = AD = 51$ units. The points E and F are on side AD so that $AE = FD = 3$ units. From which point of the broken line $AB-BC-CD$ is the segment EF seen at the largest angle?

Solution: Place the rectangle in a coordinate system: $A(0, 0)$, $D(0, 51)$, $B(38, 0)$, $C(38, 51)$; then $E(0, 3)$ and $F(0, 48)$.

On a line that does not intersect the segment EF , the segment is seen at the largest angle from the point where a circle through E and F is tangent to the line (any other point of the line is outside this circle, and so it sees EF at a smaller angle).

On AB : if the circle touches AB at P , then by the tangent-secant theorem from A , $AP^2 = AE \cdot AF = 3 \cdot 48 = 144$, so $AP = 12$, and $P(12, 0)$ is on the side AB . The angle is

$$\angle EPF = \angle APF - \angle APE, \quad \tan \angle EPF = \frac{\frac{48}{12} - \frac{3}{12}}{1 + \frac{48}{12} \cdot \frac{3}{12}} = \frac{15/4}{2} = \frac{15}{8}$$

so $\angle EPF \approx 61.93^\circ$. By symmetry, the point $(12, 51)$ of CD (at distance 12 from D) gives the same angle.

On BC : by symmetry, the best point is the midpoint $(38, 25.5)$, and the angle there is $2 \arctan \frac{22.5}{38} \approx 61.26^\circ$, which is smaller.

So EF is seen at the largest angle from the point of AB at distance 12 from A and from the point of CD at distance 12 from D ; this angle is $\arctan \frac{15}{8} \approx 61.93^\circ$.

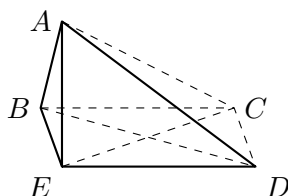
7. The points A, B, C, D, E form a pyramid whose base is the rhombus $BCDE$ and whose apex is A , and the edge AE is perpendicular to the plane $BCDE$. We know that $AE = 6$, $BC = 8$, and $\angle BCD = 120^\circ$.

a) What angle does the edge AC make with the plane of the base?

b) What is the angle between the planes ACB and ACD ?

c) How far is the edge AC from the segment BD ?

Solution: In the rhombus, $\angle BCD = 120^\circ$, so the angle at E is also 120° , and the angles at B and D are 60° . So triangles BCE and DCE are equilateral, the diagonal EC is 8, and the diagonal BD is $2 \cdot 4\sqrt{3} = 8\sqrt{3}$.



a) The projection of AC onto the base is $EC = 8$, and $AE = 6$, so the angle φ satisfies $\tan \varphi = \frac{6}{8}$, $\varphi \approx 36.87^\circ$. (Also, $AC = \sqrt{36 + 64} = 10$.)

b) Since $EB = EC = ED = 8$, we get $AB = AC = AD = 10$. So triangles ABC and ADC are congruent, with sides 10, 10, 8, and the perpendiculars from B and D to AC have the same foot H . In triangle ABC , $\cos \angle ACB = \frac{8^2 + 10^2 - 10^2}{2 \cdot 8 \cdot 10} = 0.4$, so

$$BH = DH = 8 \sin \angle ACB = 8\sqrt{1 - 0.16} = 8\sqrt{0.84}, \quad BH^2 = 53.76$$

The dihedral angle δ at edge AC is the angle BHD . By the Law of Cosines in triangle BHD ,

$$\cos \delta = \frac{BH^2 + DH^2 - BD^2}{2BH \cdot DH} = \frac{2 \cdot 53.76 - 192}{2 \cdot 53.76} = -\frac{11}{14}$$

so $\delta \approx 141.79^\circ$, and the angle between the two planes is $180^\circ - \delta \approx 38.21^\circ$.

c) The diagonals of the rhombus are perpendicular, so $BD \perp EC$; also $BD \perp AE$, since AE is perpendicular to the base. So BD is perpendicular to the plane AEC , and it intersects this plane at the center O of the rhombus. So the distance between BD and AC is the distance of O from the line AC in the right triangle AEC . Since $OC = 4$ and $\sin \angle ACE = \frac{AE}{AC} = \frac{6}{10}$, the distance is

$$OC \cdot \sin \angle ACE = 4 \cdot 0.6 = 2.4 \text{ units.}$$

8. Let $f(x) = x^2 + x + 2$ and $g(x) = x^2 - x + 2$. Prove that there is no function h , defined for every x , for which

$$h(f(x)) + h(g(x)) = g(f(x))$$

for every x .

Solution: Notice that $f(-x) = x^2 - x + 2 = g(x)$ and $g(-x) = f(x)$. Suppose that such a function h exists. Writing $-x$ in place of x ,

$$h(g(x)) + h(f(x)) = g(g(x))$$

The left-hand side is the same as in the original equation, so $g(f(x)) = g(g(x))$ for every x . But with $x = 1$: $f(1) = 4$, $g(1) = 2$, and

$$g(f(1)) = g(4) = 14 \neq 4 = g(2) = g(g(1))$$

This is a contradiction, so there is no such function h .