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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given system of equations.

$$xy(x^2 + y^2) = -742, \quad x^4 + y^4 = 2417$$

Solution: Let $p = xy$ and $s = x^2 + y^2$. Then $x^4 + y^4 = s^2 - 2p^2$, so the system is

$$ps = -742, \quad s^2 - 2p^2 = 2417$$

Substituting $s = -\frac{742}{p}$ and multiplying by p^2 :

$$\begin{aligned} 742^2 - 2p^4 &= 2417p^2 \\ 2p^4 + 2417p^2 - 550564 &= 0 \\ p^2 &= \frac{-2417 \pm \sqrt{2417^2 + 8 \cdot 550564}}{4} = \frac{-2417 \pm 3201}{4} \end{aligned}$$

Since $p^2 \geq 0$, $p^2 = 196$ and $p = \pm 14$. Since $s = x^2 + y^2 > 0$ and $ps < 0$, we need $p = -14$, and then $s = 53$. So

$$(x + y)^2 = s + 2p = 25, \quad (x - y)^2 = s - 2p = 81$$

so $x + y = \pm 5$ and $x - y = \pm 9$. The four solutions are

$$(7, -2), \quad (-2, 7), \quad (2, -7), \quad (-7, 2)$$

2. The equations of two side lines of a triangle are $5x - 13y - 26 = 0$ and $4x + 3y + 27 = 0$. Find the equation of the line that passes through the point $P(1, 3)$ and forms a triangle of the smallest possible area with the two given lines.

Solution: Let V be the intersection of the two given lines. We claim that the triangle has the smallest area when P is the midpoint of the segment QR that the two lines cut from the line through P .

Indeed, let QR be such a segment, with midpoint P , and let $Q'R'$ be any other segment through P , with Q' on line VQ and R' on line VR . Say Q' is farther from V than Q (the other case is the same). The reflection about P takes R to Q , the line VR to the line through Q parallel to it, and R' to a point R'' of this parallel line, inside triangle PQQ' . So

$$T_{VQ'R'} - T_{VQR} = T_{PQQ'} - T_{PRR'} = T_{PQQ'} - T_{PQR''} > 0$$

Now we find Q and R with midpoint P : $Q(q_1, q_2)$ is on the first line, and $R = (2 - q_1, 6 - q_2)$ is on the second one:

$$5q_1 - 13q_2 = 26, \quad 4(2 - q_1) + 3(6 - q_2) + 27 = 0, \text{ that is, } 4q_1 + 3q_2 = 53$$

Solving, $q_1 = \frac{767}{67}$ and $q_2 = \frac{161}{67}$. The slope of PQ is

$$\frac{\frac{161}{67} - 3}{\frac{767}{67} - 1} = \frac{161 - 201}{767 - 67} = -\frac{40}{700} = -\frac{2}{35}$$

so the line is $y - 3 = -\frac{2}{35}(x - 1)$, that is,

$$2x + 35y - 107 = 0$$

(The area of the triangle is $\frac{321\,600}{4489} \approx 71.64$ square units.)

3. The faces of a tetrahedron are congruent triangles, and its edges are 10 , $3\sqrt{13}$, and $\sqrt{145}$ units long. Find the volume of the tetrahedron.

Solution: A tetrahedron with congruent faces has opposite edges equal, and it can be placed in a rectangular box so that its edges are face diagonals of the box. If the edges of the box are a , b , c , then

$$a^2 + b^2 = 100, \quad b^2 + c^2 = 117, \quad a^2 + c^2 = 145$$

Adding, $a^2 + b^2 + c^2 = 181$, so $c^2 = 81$, $a^2 = 64$, $b^2 = 36$, that is, $a = 8$, $b = 6$, $c = 9$. The tetrahedron is what remains of the box after we cut off four corner pyramids, each with volume $\frac{1}{6}abc$. So

$$V = abc - 4 \cdot \frac{abc}{6} = \frac{abc}{3} = \frac{8 \cdot 6 \cdot 9}{3} = 144 \text{ cubic units.}$$

4. Solve the given equation.

$$\frac{\log_x (x^2 + x + 3)}{4x^2 + 7} + \frac{4x^2 + 7}{\log_x (x^2 + x + 3)} = \frac{2x^2 + 1}{x^2 + 1}$$

Solution: We need $x > 0$, $x \neq 1$, and $\log_x (x^2 + x + 3) \neq 0$. Let $t = \frac{\log_x (x^2 + x + 3)}{4x^2 + 7} \neq 0$. The left-hand side is $t + \frac{1}{t}$, and

$$\left| t + \frac{1}{t} \right| = |t| + \frac{1}{|t|} \geq 2$$

since t and $\frac{1}{t}$ have the same sign. But the right-hand side is

$$\frac{2x^2 + 1}{x^2 + 1} = 2 - \frac{1}{x^2 + 1}$$

which is between 1 and 2 (and never 2). So the two sides are never equal, and the equation has no solution.

5. In a chess tournament with masters and grandmasters, everyone plays everyone else twice, once with white and once with black. The total score of the masters is 1.2 times the total score of the grandmasters. How many grandmasters and masters took part in the tournament if there were three times as many masters as grandmasters?

Solution: Let g be the number of grandmasters; then there are $3g$ masters and $4g$ players. Every pair plays 2 games, and each game gives 1 point in total, so the total score is

$$2 \cdot \frac{4g(4g-1)}{2} = 4g(4g-1)$$

The masters scored 1.2 times as much as the grandmasters, so the grandmasters scored $\frac{4g(4g-1)}{2.2} = \frac{20}{11}g(4g-1)$, and the masters $\frac{24}{11}g(4g-1)$.

The masters scored at least as much as the points of their games against each other, which is $3g(3g-1)$:

$$\frac{24}{11}g(4g-1) \geq 3g(3g-1), \quad 96g-24 \geq 99g-33, \quad g \leq 3$$

Every score is a multiple of $\frac{1}{2}$, so $\frac{40}{11}g(4g-1)$ must be an integer, that is, 11 divides $g(4g-1)$. For $g = 1, 2, 3$ we get 3, 14, 33; only $g = 3$ works.

So there were 3 grandmasters and 9 masters. (Then the grandmasters scored 60 and the masters 72 points; the masters scored $72 = 9 \cdot 8$ points only in their games against each other, so the grandmasters won all of their games against the masters.)

6. Solve the given inequality.

$$x + 21 - 30\sqrt{\frac{x+21}{x-3}} - \frac{175}{x-3} \geq 0$$

For which values of x does equality hold?

Solution: (In the original, the first denominator is printed as $y - 3$; this is evidently a misprint for $x - 3$.) We need $\frac{x+21}{x-3} \geq 0$ and $x \neq 3$, that is, $x \leq -21$ or $x > 3$. Let $w = \sqrt{(x+21)(x-3)} \geq 0$.

Case 1. $x > 3$. Then $(x-3)\sqrt{\frac{x+21}{x-3}} = w$, and multiplying by $x-3 > 0$:

$$w^2 - 30w - 175 \geq 0, \quad (w-35)(w+5) \geq 0, \quad w \geq 35$$

So $(x+21)(x-3) \geq 1225$, that is, $x^2 + 18x - 1288 = (x-28)(x+46) \geq 0$, and with $x > 3$: $x \geq 28$.

Case 2. $x \leq -21$. Then $x-3 < 0$ and $(x-3)\sqrt{\frac{x+21}{x-3}} = -w$. Multiplying by $x-3 < 0$ reverses the inequality:

$$w^2 + 30w - 175 \leq 0, \quad (w+35)(w-5) \leq 0, \quad w \leq 5$$

So $(x+21)(x-3) \leq 25$, that is, $x^2 + 18x - 88 = (x+22)(x-4) \leq 0$, and with $x \leq -21$: $-22 \leq x \leq -21$.

So the solution is $-22 \leq x \leq -21$ or $x \geq 28$, and equality holds for $x = -22$ and $x = 28$.

7. Solve the given equation.

$$\sqrt{\frac{(1-\sqrt{3})(2\cos x + 1)}{8}} + \cos \frac{x}{2} = 0$$

Solution: Since $1 - \sqrt{3} < 0$, the square root is defined only if $2\cos x + 1 \leq 0$, that is, $\cos x \leq -\frac{1}{2}$. Also, the square root is non-negative, so $\cos \frac{x}{2} \leq 0$. Squaring, and using $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$, with $c = \cos x$:

$$\begin{aligned} \frac{(1-\sqrt{3})(2c+1)}{8} &= \frac{1+c}{2} && \text{multiply by 8} \\ 2c - 2\sqrt{3}c + 1 - \sqrt{3} &= 4 + 4c \\ -(2 + 2\sqrt{3})c &= 3 + \sqrt{3} \\ c &= -\frac{3 + \sqrt{3}}{2(1 + \sqrt{3})} = -\frac{\sqrt{3}}{2} \end{aligned}$$

(The last step: $3 + \sqrt{3} = \sqrt{3}(\sqrt{3} + 1)$.) So $\cos x = -\frac{\sqrt{3}}{2}$, and we also need $\cos \frac{x}{2} < 0$. Then $\cos^2 \frac{x}{2} = \frac{2 - \sqrt{3}}{4}$,

so $\cos \frac{x}{2} = -\frac{\sqrt{6} - \sqrt{2}}{4} = \cos 105^\circ$, and $\frac{x}{2} = \pm 105^\circ + k \cdot 360^\circ$. So the solutions are

$$x = \pm \frac{7\pi}{6} + 4k\pi \quad \text{where } k \text{ is any integer.}$$

8. Solve the given inequality.

$$\sqrt{\frac{(1 - \sqrt{3})(2 \cos x + 1)}{8}} + \cos \frac{x}{2} \geq 0$$

Solution: The left-hand side has period 4π , so we work on $[0, 4\pi)$. As in the previous problem, the square root is defined only if $\cos x \leq -\frac{1}{2}$, that is, $x \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ or $x \in \left[\frac{8\pi}{3}, \frac{10\pi}{3}\right]$.

If $\cos \frac{x}{2} \geq 0$, the inequality holds. On $[0, 4\pi)$ this happens for $x \in [0, \pi]$ and $x \in [3\pi, 4\pi)$.

If $\cos \frac{x}{2} < 0$, we need $\sqrt{\dots} \geq -\cos \frac{x}{2}$; squaring and following the computation of the previous problem (dividing by the negative number $-(2 + 2\sqrt{3})$ reverses the inequality), this means $\cos x \leq -\frac{\sqrt{3}}{2}$, that is, $x \in \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right]$ or $x \in \left[\frac{17\pi}{6}, \frac{19\pi}{6}\right]$.

Putting the pieces together on $[0, 4\pi)$: $\left[\frac{2\pi}{3}, \pi\right]$ and $\left(\pi, \frac{7\pi}{6}\right]$ from the first domain interval, and $\left[\frac{17\pi}{6}, 3\pi\right)$ and $\left[3\pi, \frac{10\pi}{3}\right]$ from the second one. So the solution is

$$\frac{2\pi}{3} + 4k\pi \leq x \leq \frac{7\pi}{6} + 4k\pi \quad \text{or} \quad -\frac{7\pi}{6} + 4k\pi \leq x \leq -\frac{2\pi}{3} + 4k\pi$$

where k is any integer.