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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation over the real numbers.

$$3 \log_{1000} (x - 1) + 2 \log_{100} (x + 3) - 2^{x+3} \cdot \log_{10} (x^2 + 2x - 3) = 0$$

Solution: We need $x - 1 > 0$, $x + 3 > 0$, and $x^2 + 2x - 3 = (x - 1)(x + 3) > 0$, that is, $x > 1$. We write every logarithm with base 10, using $\log_{1000} t = \frac{\log t}{3}$ and $\log_{100} t = \frac{\log t}{2}$.

(In the original, the second term is printed as $2 \log_{1000} (x + 3)$. With that term, the equation has only one solution, $x \approx 1.2301$, which can be found only numerically, so this is evidently a misprint; most likely the base of this logarithm should be 100.)

The equation becomes

$$\log (x - 1) + \log (x + 3) - 2^{x+3} \log ((x - 1)(x + 3)) = 0, \quad \text{that is,} \quad (1 - 2^{x+3}) \log (x^2 + 2x - 3) = 0$$

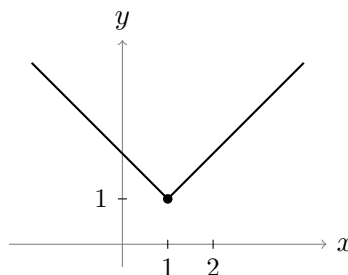
The first factor is zero only if $x = -3$, which is not in the domain. So $\log (x^2 + 2x - 3) = 0$, that is, $x^2 + 2x - 4 = 0$, and $x = -1 \pm \sqrt{5}$. Only $x = \sqrt{5} - 1 \approx 1.236$ is in the domain.

2. Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows: for every real number x , $f(x)$ is the distance of x from the integer in the closed interval $[0, 2]$ that is farthest from x . Find a formula for f and graph it.

Solution: The integers in $[0, 2]$ are 0, 1, 2. The farthest one from x is 2 if $x \leq 1$, and 0 if $x \geq 1$. So

$$f(x) = \begin{cases} 2 - x & \text{if } x \leq 1 \\ x & \text{if } x \geq 1 \end{cases} \quad \text{that is,} \quad f(x) = |x - 1| + 1$$

The graph is a "V" with its vertex at (1, 1).



3. The hypotenuse of a right triangle is c , and its legs are a and b . We rotate the triangle around each of its sides. Let V_c be the volume of the solid we get by rotating around the hypotenuse, and V_a, V_b the volumes of the solids we get by rotating around the legs. Prove that

$$\frac{1}{V_c^2} = \frac{1}{V_a^2} + \frac{1}{V_b^2}$$

Solution: Rotating around a leg gives a cone. Around leg a , the radius is b and the height is a ; around leg b , the radius is a and the height is b :

$$V_a = \frac{\pi b^2 a}{3}, \quad V_b = \frac{\pi a^2 b}{3}$$

Rotating around the hypotenuse gives two cones with a common base, whose radius is the altitude $m = \frac{ab}{c}$ belonging to the hypotenuse, and whose heights add up to c :

$$V_c = \frac{\pi m^2 c}{3} = \frac{\pi a^2 b^2}{3c}$$

Then

$$\frac{1}{V_a^2} + \frac{1}{V_b^2} = \frac{9}{\pi^2} \left(\frac{1}{a^2 b^4} + \frac{1}{a^4 b^2} \right) = \frac{9}{\pi^2} \cdot \frac{a^2 + b^2}{a^4 b^4} = \frac{9c^2}{\pi^2 a^4 b^4} = \frac{1}{V_c^2}$$

4. Which real numbers x satisfy the given equation?

$$\sin^3 x (1 + \cot x) + \cos^3 x (1 + \tan x) = \cos 2x$$

Solution: We need $\sin x \neq 0$ and $\cos x \neq 0$. The left-hand side is

$$\begin{aligned} & \sin^3 x + \sin^2 x \cos x + \cos^3 x + \cos^2 x \sin x \\ = & \sin^2 x (\sin x + \cos x) + \cos^2 x (\cos x + \sin x) = \sin x + \cos x \end{aligned}$$

and $\cos 2x = (\cos x - \sin x)(\cos x + \sin x)$. So

$$(\sin x + \cos x)(1 - \cos x + \sin x) = 0$$

If $\sin x + \cos x = 0$, then $\tan x = -1$, and $x = -\frac{\pi}{4} + k\pi$.

If $\cos x - \sin x = 1$, then $\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) = 1$, so $x = 2k\pi$ or $x = -\frac{\pi}{2} + 2k\pi$. But then $\sin x = 0$ or $\cos x = 0$, which is not allowed.

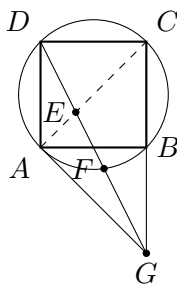
So the solutions are $x = -\frac{\pi}{4} + k\pi$, where k is any integer.

5. The side of square $ABCD$ is a . We choose a point E on the diagonal AC so that $AE : AC = 1 : 3$. The line DE intersects the circle circumscribed around the square at F , and the extension of side CB at G . Prove that

a) GA is tangent to the circle;

b) $GF \cdot GD = 2a^2$.

Solution: We use coordinates: $A(0, 0)$, $B(a, 0)$, $C(a, a)$, $D(0, a)$. Then $E = \left(\frac{a}{3}, \frac{a}{3}\right)$. The line DE goes through $(0, a)$ with slope $\frac{a/3 - a}{a/3} = -2$, so its equation is $y = a - 2x$. The line CB is $x = a$, so $G = (a, -a)$.



a) The center of the circle is $O\left(\frac{a}{2}, \frac{a}{2}\right)$. Since $\overrightarrow{GA} = (-a, a)$ and $\overrightarrow{OA} = \left(-\frac{a}{2}, -\frac{a}{2}\right)$, their dot product is $\frac{a^2}{2} - \frac{a^2}{2} = 0$. So GA is perpendicular to the radius OA at its endpoint, and it is tangent to the circle.

b) By the tangent-secant theorem from G , $GF \cdot GD = GA^2 = a^2 + a^2 = 2a^2$.

6. In the tetrahedron $ABCS$, the edges SA, SB, SC are pairwise perpendicular. Is triangle ABC acute?

Solution: Yes. Let $SA = p, SB = q, SC = r$. By the Pythagorean theorem,

$$AB^2 = p^2 + q^2, \quad BC^2 = q^2 + r^2, \quad CA^2 = r^2 + p^2$$

By the Law of Cosines, the angle at A satisfies

$$\cos \angle A = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC} = \frac{2p^2}{2 \cdot AB \cdot AC} > 0$$

so the angle at A is acute. The same way, the angles at B and C are acute, so triangle ABC is acute.

7. Starting with some integer, we take every eighth integer until the sum of the numbers taken is 473. Which number did we take first, and how many numbers did we take?

Solution: Let the first number be a , and let us take n numbers: $a, a + 8, \dots, a + 8(n - 1)$. Their sum is

$$na + 8 \cdot \frac{n(n-1)}{2} = n(a + 4(n-1)) = 473 = 11 \cdot 43$$

So n is a positive divisor of 473, that is, 1, 11, 43, or 473, and $a = \frac{473}{n} - 4(n-1)$:

- $n = 1$: $a = 473$ (we take only the number 473);
- $n = 11$: $a = 43 - 40 = 3$ (the numbers 3, 11, 19, $\dots$, 83);
- $n = 43$: $a = 11 - 168 = -157$;
- $n = 473$: $a = 1 - 1888 = -1887$.

If we start with a positive integer and take more than one number, the only possibility is to start with 3 and take 11 numbers. (Check: $\frac{11(3+83)}{2} = 473$.)

8. Find the integer values of the parameter k for which the roots of the equation

$$(k^2 - 1)x^2 - (2k^2 - 3k - 1)x + k^2 - 3k + 2 = 0$$

are integers.

Solution: The left-hand side factors:

$$((k - 1)x - (k - 2))((k + 1)x - (k - 1)) = 0$$

Check: the coefficient of x is

$$-((k - 1)^2 + (k - 2)(k + 1)) = -(2k^2 - 3k - 1)$$

and the constant term is $(k - 1)(k - 2) = k^2 - 3k + 2$.

If $k = 1$: the equation is $2x = 0$, with the integer root $x = 0$.

If $k = -1$: the equation is $-4x + 6 = 0$, with root $\frac{3}{2}$, not an integer.

If $k \neq \pm 1$, the roots are

$$x_1 = \frac{k - 2}{k - 1} = 1 - \frac{1}{k - 1}, \quad x_2 = \frac{k - 1}{k + 1} = 1 - \frac{2}{k + 1}$$

x_1 is an integer only if $k - 1 = \pm 1$, that is, $k = 0$ or $k = 2$; x_2 is an integer only if $k + 1$ divides 2, that is, $k \in \{-3, -2, 0, 1\}$. Both are integers only if $k = 0$, and then the roots are 2 and -1 .

So $k = 0$ (roots 2 and -1), and also $k = 1$ if we allow the equation to become linear (root 0).