translated from: KoMaL, November 1987, pages 361-362
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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. Find the exact value of the given expression without using a calculator or tables.

$$\frac{5 \cdot 8^{-\frac{2}{3}} + \frac{2}{\sqrt{3}} \left(\sin \frac{16\pi}{3} \right)^3}{\log_4 2}$$

(9 points)

Solution: $8^{-\frac{2}{3}} = \left(2^3\right)^{-\frac{2}{3}} = \frac{1}{4}$. Since $\frac{16\pi}{3} = 4\pi + \frac{4\pi}{3}$, $\sin \frac{16\pi}{3} = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, and its cube is $-\frac{3\sqrt{3}}{8}$. Also $\log_4 2 = \frac{1}{2}$. So the value is

$$\frac{\frac{5}{4} + \frac{2}{\sqrt{3}} \cdot \left(-\frac{3\sqrt{3}}{8} \right)}{\frac{1}{2}} = \frac{\frac{5}{4} - \frac{3}{4}}{\frac{1}{2}} = 1$$

2. The vertices A and B of parallelogram $ABCD$ are $A\,(1, 4)$ and $B\,(6, 6)$. The point $P\,(10, 18)$ is on the line BC, and the point $R\,(-1, 11)$ is on the line CD. Find the perimeter of the parallelogram. (11 points)

Solution: The line BC goes through $B\,(6, 6)$ and $P\,(10, 18)$, with direction $(4, 12)$, which is parallel to $(1, 3)$. So its points are $(6 + t, 6 + 3t)$. The line CD is parallel to AB, whose direction is $(5, 2)$, and goes through R: its points are $(-1 + 5s, 11 + 2s)$. At C:

$$6 + t = -1 + 5s, \qquad 6 + 3t = 11 + 2s$$

so $t = 5s - 7$, $13s = 26$, $s = 2$, $t = 3$, and $C = (9, 15)$. So $AB = \sqrt{29}$ and $BC = \sqrt{9 + 81} = 3\sqrt{10}$, and the perimeter is

$$2\left(\sqrt{29} + 3\sqrt{10}\right) \approx 29.74$$

3. The base of an isosceles triangle is 36 units long, and its legs are 30 units long. How far is the centroid of the triangle from the center of its circumscribed circle? (12 points)

Solution: Place the base on the x-axis: $(-18, 0)$, $(18, 0)$, and the apex at $(0, h)$ with $h = \sqrt{30^2 - 18^2} = 24$. The centroid is $(0, 8)$. The center of the circumscribed circle is $(0, k)$ with

$$18^2 + k^2 = (24 - k)^2, \qquad 324 = 576 - 48k, \qquad k = \frac{21}{4}$$

So the distance is $8 - \frac{21}{4} = \frac{11}{4}$.

4. Find the largest subset of the real numbers on which the following expressions are defined.

   a) $\dfrac{\sqrt{x^2+4x-5}}{|x+2|-3}$ (8 points)

   b) $\log(1-\sin 2x)$ (5 points)

   Solution: a) We need $x^2+4x-5=(x+5)(x-1)\geq 0$, that is, $x\leq -5$ or $x\geq 1$, and $|x+2|\neq 3$, that is, $x\neq 1$ and $x\neq -5$. So the expression is defined for $x<-5$ and for $x>1$.

   b) We need $1-\sin 2x>0$, that is, $\sin 2x\neq 1$, so $2x\neq \dfrac{\pi}{2}+2k\pi$. The expression is defined for every real x except $x=\dfrac{\pi}{4}+k\pi$.

5. The base of a triangular pyramid is an equilateral triangle, and its lateral faces are congruent isosceles triangles. The cross-section through one lateral edge, perpendicular to the opposite base edge, has an area of $150\,\mathrm{cm}^2$. The volume of the pyramid is $1500\,\mathrm{cm}^3$. What angle does each lateral face make with the base? (13 points)

   Solution: Let the base be the equilateral triangle ABC with side a, the apex D, and the height m. The apex is above the center O of the base. The cross-section through DA perpendicular to BC is the triangle ADF, where F is the midpoint of BC; its base is $AF=\dfrac{\sqrt{3}}{2}a$ and its height is m. So

$$\frac{1}{2}\cdot\frac{\sqrt{3}}{2}am=150,\qquad V=\frac{1}{3}\cdot\frac{\sqrt{3}}{4}a^2m=1500$$

   Dividing the second equation by the first one: $\dfrac{a}{3}=10$, so $a=30$, and then $m=\dfrac{600}{\sqrt{3}\cdot 30}=\dfrac{20\sqrt{3}}{3}$.

   The angle φ between the face DBC and the base is the angle DFO, where $OF=\dfrac{\sqrt{3}}{6}a=5\sqrt{3}$. So

$$\tan\varphi=\frac{m}{OF}=\frac{20\sqrt{3}/3}{5\sqrt{3}}=\frac{4}{3},\qquad \varphi\approx 53.13^\circ$$

Recommended for students from academic high schools

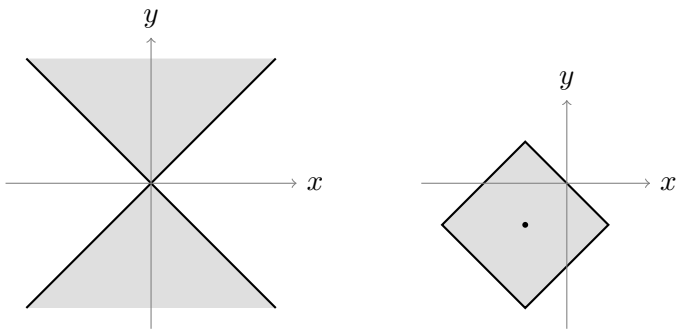
6. Sketch the set of points $P(x, y)$ in the coordinate plane whose coordinates satisfy the following conditions.

   a) $|x| \leq |y|$ (5 points)

   b) $|x+1| + |y+1| \leq 2$ (8 points)

   Solution: a) The condition $|x| \leq |y|$ describes the points that are at most as far from the y-axis as from the x-axis (since $|x|$ is the distance from the y-axis and $|y|$ is the distance from the x-axis): the region between the lines $y = x$ and $y = -x$ that contains the y-axis, including the two lines.

   b) The condition says that the sum of the distances of (x, y) from the lines $x = -1$ and $y = -1$ is at most 2. This is the square (with its interior) whose vertices are $(1, -1)$, $(-1, 1)$, $(-3, -1)$, $(-1, -3)$, centered at $(-1, -1)$.



a) b)

7. Solve the given system of equations over the real numbers.

$$\begin{aligned} x^{\log_8 y} + y^{\log_8 x} &= 4 \\ \log_4 x - \log_4 y &= 1 \end{aligned}$$

(14 points)

Solution: We need $x > 0$ and $y > 0$. Notice that $x^{\log_8 y} = 8^{\log_8 x \cdot \log_8 y} = y^{\log_8 x}$, so the first equation says $2x^{\log_8 y} = 4$, that is, $x^{\log_8 y} = 2$. Let $u = \log_8 x$ and $v = \log_8 y$. Then the first equation is $8^{uv} = 2$, so $uv = \dfrac{1}{3}$. The second equation says $\dfrac{x}{y} = 4$, so $u - v = \log_8 4 = \dfrac{2}{3}$. Then

$$u\left(u - \frac{2}{3}\right) = \frac{1}{3}, \qquad 3u^2 - 2u - 1 = 0, \qquad u = 1 \text{ or } u = -\frac{1}{3}$$

If $u = 1$: $v = \dfrac{1}{3}$, so $x = 8$, $y = 2$. If $u = -\dfrac{1}{3}$: $v = -1$, so $x = \dfrac{1}{2}$, $y = \dfrac{1}{8}$. (Check: $8^{\log_8 2} + 2^{\log_8 8} = 2 + 2 = 4$, and $\left(\dfrac{1}{2}\right)^{-1} + \left(\dfrac{1}{8}\right)^{-\frac{1}{3}} = 2 + 2 = 4$.)

8. Let O be a point in the plane, and let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be unit vectors such that O is outside the triangle ABC. Prove that

$$\left|\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right|>1$$

(15 points)

Solution: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the three unit vectors. Since O is outside the triangle ABC, there is a line through O such that the whole triangle is on one side of it. So the points A, B, C of the unit circle around O are on an open half circle. Let B be the middle one of the three points on this half circle, and let $\alpha=\angle AOB$ and $\gamma=\angle BOC$; then $\alpha+\gamma<180^\circ$. Now

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{b}=\cos\alpha+1+\cos\gamma$$

Since $\alpha<180^\circ-\gamma$, $\cos\alpha>\cos(180^\circ-\gamma)=-\cos\gamma$, so $\cos\alpha+\cos\gamma>0$. So

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}|\geq(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{b}>1$$

(the length of a vector is at least its dot product with a unit vector).

Recommended for students from vocational secondary schools

6. Solve the given inequality over the real numbers.

$$2\log_x\frac{1}{4}+\log_4\frac{1}{x}\leq-3$$

(13 points)

Solution: We need $x>0$ and $x\neq1$. Let $t=\log_4 x\neq0$. Then $\log_x\frac{1}{4}=-\frac{1}{t}$ and $\log_4\frac{1}{x}=-t$, so

$$-\frac{2}{t}-t\leq-3,\qquad t+\frac{2}{t}-3\geq0,\qquad\frac{(t-1)(t-2)}{t}\geq0$$

A sign chart gives $0<t\leq1$ or $t\geq2$, that is, $1<x\leq4$ or $x\geq16$.

7. In triangle ABC, the median AA_1 is perpendicular to AB. Let a, b, c be the lengths of the sides BC, CA, AB. Express the cosine of the angle CAB in terms of b and c, and the cosine of the angle BCA in terms of a and b. (14 points)

Solution: Use vectors from A: $\overrightarrow{AA_1}=\dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2}$. Since $AA_1\perp AB$,

$$\left(\overrightarrow{AB}+\overrightarrow{AC}\right)\cdot\overrightarrow{AB}=0,\qquad c^2+bc\cos\angle CAB=0,\qquad\cos\angle CAB=-\frac{c}{b}$$

By the Law of Cosines, $a^2=b^2+c^2-2bc\cos\angle CAB=b^2+3c^2$, so $c^2=\dfrac{a^2-b^2}{3}$. Then

$$\cos\angle BCA=\frac{a^2+b^2-c^2}{2ab}=\frac{a^2+b^2-\frac{a^2-b^2}{3}}{2ab}=\frac{a^2+2b^2}{3ab}$$

8. Let a and b be real numbers with $a > b$ and $ab = 1$. Prove that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2}$$

(15 points)

Solution: Since $ab = 1$, $a^2 + b^2 = (a - b)^2 + 2ab = (a - b)^2 + 2$, so with $d = a - b > 0$,

$$\frac{a^2 + b^2}{a - b} = d + \frac{2}{d} \geq 2\sqrt{d \cdot \frac{2}{d}} = 2\sqrt{2}$$

by the inequality between the arithmetic and geometric means.

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