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These problems were given in 1987 at the written mathematics entrance examination of Hungarian students applying to study at universities abroad. Problems 1–5 were the same for everyone. Of problems 6–8, the ones marked A were recommended for students who graduated from academic high schools, and the ones marked B for students who graduated from vocational secondary schools.

1. The numerators of two fractions are both 3, and the sum of their denominators is 13. The sum of the two fractions is $\frac{39}{40}$. Find the two fractions.

   Solution: Let the denominators be p and q. Then $p+q=13$ and

   $$\frac{3}{p}+\frac{3}{q}=\frac{3\,(p+q)}{pq}=\frac{39}{pq}=\frac{39}{40}, \qquad \text{so} \qquad pq=40$$

   So p and q are the roots of $t^2-13t+40=0$, that is, 5 and 8. The fractions are $\frac{3}{5}$ and $\frac{3}{8}$.

2. Solve the given system of equations over the real numbers.

   $$7^{x+1}\cdot 2^y=4, \qquad \frac{1}{x-1}-\frac{1}{y}=\frac{4}{y\,(x-1)}$$

   Solution: We need $x\neq 1$ and $y\neq 0$. Multiplying the second equation by $y\,(x-1)$: $y-(x-1)=4$, so $y=x+3$. Substituting into the first equation,

   $$7^{x+1}\cdot 2^{x+3}=4, \qquad 7^{x+1}\cdot 2^{x+1}\cdot 4=4, \qquad 14^{x+1}=1$$

   So $x=-1$ and $y=2$.

3. The diagonals of an isosceles trapezoid are perpendicular, and they divide each other into two parts in a $1:7$ ratio. Find the area of the trapezoid if its perimeter is $180\sqrt{2}$.

   Solution: Let M be the intersection of the diagonals. By symmetry, the diagonals are equal and M divides both in the same ratio, so $MA=MB=7k$ and $MC=MD=k$ for some k. Since the diagonals are perpendicular, the triangles around M are right triangles, so

   $$AB=7\sqrt{2}k, \quad CD=\sqrt{2}k, \quad AD=BC=\sqrt{49k^2+k^2}=5\sqrt{2}k$$

   The perimeter is $18\sqrt{2}k=180\sqrt{2}$, so $k=10$, and each diagonal is $8k=80$. The area of a quadrilateral with perpendicular diagonals is half of their product:

   $$T=\frac{80\cdot 80}{2}=3200 \text{ square units.}$$

4. Two circles with radii R and r ($R > r$) are tangent to each other from outside. Express, as a function of the radii, the length of the segment of their common internal tangent line that lies between the two common external tangent lines.

   Solution: Let A be the point of tangency of the circles, and let one external tangent line touch the circles at E_1 and E_2. The internal tangent line at A meets E_1E_2 at a point M. The tangent segments from M are equal: $ME_1 = MA = ME_2$, so $MA = \dfrac{E_1E_2}{2}$.

   The length of the external tangent segment: the centers are $R + r$ apart, and the radii to E_1 and E_2 are perpendicular to E_1E_2, so

$$E_1E_2 = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$$

   So $MA = \sqrt{Rr}$. The same holds for the other external tangent line, on the other side of A. So the segment of the internal tangent between the two external tangents has length $2\sqrt{Rr}$.

5. The sum of the first, third, and fifth terms of a geometric sequence is 182, and the sum of the second and fourth terms is 20 times the common ratio. Write down the first five terms of the sequence.

   Solution: Let the first term be a and the ratio q (with $a \neq 0$, $q \neq 0$). Then

$$a\left(1 + q^2 + q^4\right) = 182, \qquad aq\left(1 + q^2\right) = 20q, \text{ so } a\left(1 + q^2\right) = 20$$

   Dividing, with $u = q^2$:

$$\frac{1 + u + u^2}{1 + u} = \frac{182}{20}, \qquad 10u^2 - 81u - 81 = 0, \qquad u = 9 \text{ (or } u = -\frac{9}{10}\text{, which is impossible)}$$

   So $q = \pm 3$ and $a = \dfrac{20}{10} = 2$. The sequence is

$$2,\ 6,\ 18,\ 54,\ 162 \qquad \text{or} \qquad 2,\ -6,\ 18,\ -54,\ 162$$

6A. Let ABC and DEF be any two triangles, and let S and T be their centroids. Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 3\overrightarrow{ST}$.

   Solution: With position vectors, $\mathbf{s} = \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$ and $\mathbf{t} = \dfrac{\mathbf{d}+\mathbf{e}+\mathbf{f}}{3}$, so

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = (\mathbf{d}+\mathbf{e}+\mathbf{f}) - (\mathbf{a}+\mathbf{b}+\mathbf{c}) = 3\mathbf{t} - 3\mathbf{s} = 3\overrightarrow{ST}$$

7A. Solve the given inequality over the real numbers.

$$\log_{|x-1|} x \geq 1$$

   Solution: We need $x > 0$, $|x-1| > 0$, and $|x-1| \neq 1$, that is, $x > 0$, $x \neq 1$, $x \neq 2$. We write $1 = \log_{|x-1|} |x-1|$.

   If the base is greater than 1, that is, $x > 2$: we need $x \geq |x-1| = x - 1$, which is always true. So every $x > 2$ is a solution.

   If the base is between 0 and 1, that is, $0 < x < 2$ and $x \neq 1$: the inequality reverses, $x \leq |x-1|$. For $0 < x < 1$ this means $x \leq 1 - x$, so $0 < x \leq \dfrac{1}{2}$; for $1 < x < 2$ it means $x \leq x - 1$, which is false.

   So the solution is $0 < x \leq \dfrac{1}{2}$ or $x > 2$.

8A. Is there a line in the plane all of whose points $P(x, y)$ satisfy the inequality $x^4 - 2x^2y + y^2 - 1 \leq 0$?

Solution: No. The inequality is $\left(y - x^2\right)^2 \leq 1$, that is, $x^2 - 1 \leq y \leq x^2 + 1$: the region between two parabolas. A vertical line $x = c$ contains points with y arbitrarily large, which are not in the region. A non-vertical line $y = mx + k$ is not in the region either, since $x^2 - 1 - (mx + k)$ is positive for large x, so the points of the line with large x are below the lower parabola.

6B. In trapezoid $ABCD$, the base AB is twice as long as the parallel side CD and twice as long as the leg AD. The other leg BC is 5 units long, and the diagonal AC is 12 units long. Find the area of the trapezoid.

Solution: Let $CD = AD = a$, so $AB = 2a$, and let F be the midpoint of AB. Then $AF = a = DC$ and $AF \parallel DC$, so $AFCD$ is a parallelogram, and since $AD = a$, it is a rhombus: $FC = a$. So F is at distance a from A, B, and C, and by the converse of Thales' theorem, the angle at C in triangle ABC is a right angle. So

$$AB = \sqrt{12^2 + 5^2} = 13, \quad a = 6.5, \quad \text{and the height is} \quad m = \frac{AC \cdot BC}{AB} = \frac{60}{13}$$

The area is

$$T = \frac{13 + 6.5}{2} \cdot \frac{60}{13} = 45 \text{ square units.}$$

7B. Find the solutions of the following inequalities in the interval $\left[\frac{\pi}{4}, \pi\right]$.

a) $\dfrac{1}{\sin^2 x} \leq \dfrac{1}{\sin 2x - \cos^2 x}$ b) $\log_2 \sin 2x < -1$

Solution: a) We need $\sin x \neq 0$ and $D = \sin 2x - \cos^2 x \neq 0$. The left-hand side is positive. If $D < 0$, the right-hand side is negative, and the inequality is false. If $D > 0$, the inequality is equivalent to $D \leq \sin^2 x$, that is,

$$0 \leq \sin^2 x - 2 \sin x \cos x + \cos^2 x = (\sin x - \cos x)^2$$

which is always true. So the solutions are exactly the x with $D = \cos x\,(2 \sin x - \cos x) > 0$. In $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x > 0$ and $2 \sin x > \cos x$, so $D > 0$; at $\frac{\pi}{2}$, $D = 0$; in $\left(\frac{\pi}{2}, \pi\right]$, $\cos x < 0 < 2 \sin x - \cos x$ (or $\sin x = 0$ at π), so $D < 0$. The solution is $\dfrac{\pi}{4} \leq x < \dfrac{\pi}{2}$.

b) We need $0 < \sin 2x < \dfrac{1}{2}$. For $x \in \left[\frac{\pi}{4}, \pi\right]$, $2x \in \left[\frac{\pi}{2}, 2\pi\right]$, and $0 < \sin 2x < \dfrac{1}{2}$ holds for $\dfrac{5\pi}{6} < 2x < \pi$. The solution is $\dfrac{5\pi}{12} < x < \dfrac{\pi}{2}$.

8B. Let P be a point on the circle circumscribed around the equilateral triangle ABC. Prove that the sum $PA^2 + PB^2 + PC^2$ is constant, and find its value.

Solution: Let O be the center and R the radius of the circle, and use position vectors from O. Then $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = |\mathbf{p}| = R$, and since O is also the centroid, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. So

$$PA^2 + PB^2 + PC^2 = \sum (\mathbf{p} - \mathbf{a})^2 = 3R^2 + 3R^2 - 2\mathbf{p} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 6R^2$$

which is constant. If the side of the triangle is a, then $R = \dfrac{a}{\sqrt{3}}$, and the constant is $6R^2 = 2a^2$.

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