translated from: KoMaL, October 1987, pages 291-292
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These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1987. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. If the sum of 101 consecutive odd numbers is 12 827, what are the smallest and the largest of them? (9 points)

   Solution: The numbers form an arithmetic sequence with 101 terms, so their average is the middle (51st) term: $\frac{12\,827}{101} = 127$. There are 50 terms on each side, each 2 apart, so the smallest is $127 - 100 = 27$ and the largest is $127 + 100 = 227$.

2. Solve the given system of equations over the real numbers.

$$x^2 - y^2 = \frac{5}{2}, \qquad xy = \frac{\sqrt{6}}{2}$$

   (10 points)

   Solution: Let $u = x^2$ and $v = -y^2$. Then $u + v = \frac{5}{2}$ and $uv = -(xy)^2 = -\frac{3}{2}$, so u and v are the roots of $t^2 - \frac{5}{2}t - \frac{3}{2} = 0$, that is, $2t^2 - 5t - 3 = 0$: $t = 3$ and $t = -\frac{1}{2}$. Since $u \geq 0 \geq v$, $x^2 = 3$ and $y^2 = \frac{1}{2}$. Since $xy > 0$, x and y have the same sign, so

$$x = \sqrt{3},\ y = \frac{\sqrt{2}}{2} \qquad \text{or} \qquad x = -\sqrt{3},\ y = -\frac{\sqrt{2}}{2}$$

3. The reflection of the origin O about the point $A\,(5, 5)$ is O_1, the reflection of O_1 about the point B is O_2, and the reflection of O_2 about the point $C\,(1, 7)$ is O again. Find the coordinates of B, and prove that the quadrilateral $OABC$ is a rhombus. (13 points)

   Solution: The reflection about a point M takes X to $2M - X$. So $O_1 = (10, 10)$, and since the reflection of O_2 about C is O, $O_2 = 2C - O = (2, 14)$. B is the midpoint of O_1O_2: $B = (6, 12)$.

   Now $\overrightarrow{OA} = (5, 5) = \overrightarrow{CB}$, so $OABC$ is a parallelogram, and $\left|\overrightarrow{OA}\right| = \sqrt{50} = \left|\overrightarrow{OC}\right|$, so its sides are equal: it is a rhombus.

4. In the acute triangle ABC, side AB is 16 units long, and the altitude belonging to it is 12. We inscribe an isosceles right triangle in triangle ABC so that its hypotenuse is parallel to AB, its right-angle vertex is on AB, and its other two vertices are on AC and BC. Find the sides of the inscribed triangle. (13 points)

   Solution: Let the hypotenuse of the inscribed triangle be L, at distance d from AB. The right-angle vertex is on AB, and the height of an isosceles right triangle belonging to the hypotenuse is half of the hypotenuse, so $d = \frac{L}{2}$. The hypotenuse cuts off from ABC a triangle similar to it, with height $12 - d$, so

$$\frac{L}{16} = \frac{12 - d}{12} = \frac{12 - L/2}{12}, \qquad 12L = 192 - 8L, \qquad L = \frac{48}{5} = 9.6$$

   The legs are $\frac{L}{\sqrt{2}} = \frac{24\sqrt{2}}{5} \approx 6.79$. So the sides of the inscribed triangle are 9.6, $\frac{24\sqrt{2}}{5}$, and $\frac{24\sqrt{2}}{5}$.

5. Solve the given equation over the real numbers.

$$\sin x + \cos x = \sin 2x - \frac{1}{2}$$

(13 points)

Solution: Let $s = \sin x + \cos x$. Then $s^2 = 1 + \sin 2x$, so $\sin 2x = s^2 - 1$, and

$$s = s^2 - \frac{3}{2}, \qquad 2s^2 - 2s - 3 = 0, \qquad s = \frac{1 \pm \sqrt{7}}{2}$$

Since $s = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, we need $|s| \le \sqrt{2}$, and $\frac{1+\sqrt{7}}{2} \approx 1.82$ is too large. So

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1-\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14}-\sqrt{2}}{4} \approx -0.5819$$

Let $\beta = \arcsin\frac{\sqrt{14}-\sqrt{2}}{4} \approx 35.58°$. Then $x + 45° = -\beta + k \cdot 360°$ or $x + 45° = 180° + \beta + k \cdot 360°$, so

$$x \approx -80.58° + k \cdot 360° \qquad \text{or} \qquad x \approx 170.58° + k \cdot 360°$$

Recommended for students from academic high schools

6. For which x do the following expressions take positive values?

a) $2^x + 2^{-x} - \frac{17}{4}$ b) $2 + \log_{1/3}(5x - 1)$ (13 points)

Solution: a) Let $t = 2^x > 0$. Then $t + \frac{1}{t} > \frac{17}{4}$, that is, $4t^2 - 17t + 4 = (4t-1)(t-4) > 0$, so $t < \frac{1}{4}$ or $t > 4$. So $x < -2$ or $x > 2$.

b) We need $5x - 1 > 0$. The inequality is $\log_{1/3}(5x-1) > -2 = \log_{1/3} 9$, and since the base is less than 1, $5x - 1 < 9$. So $\frac{1}{5} < x < 2$.

7. In rectangle $ABCD$, side AB is three times side BC. The distances of an inside point P from the vertices B, A, D are $PB = 4\sqrt{2}$, $PA = \sqrt{2}$, and $PD = 2$. Find the area of the rectangle. (14 points)

Solution: Use coordinates: $A(0,0)$, $B(3a,0)$, $D(0,a)$, and $P(x,y)$. Then

$$x^2 + y^2 = 2, \qquad (x-3a)^2 + y^2 = 32, \qquad x^2 + (y-a)^2 = 4$$

Subtracting the first equation from the other two: $-6ax + 9a^2 = 30$ and $-2ay + a^2 = 2$, so

$$x = \frac{9a^2 - 30}{6a}, \qquad y = \frac{a^2 - 2}{2a}$$

Substituting into $x^2 + y^2 = 2$ and writing $s = a^2$: $(9s - 30)^2 + 9(s-2)^2 = 72s$, that is, $5s^2 - 36s + 52 = 0$, so $s = 2$ or $s = \frac{26}{5}$. If $s = 2$, then $x < 0$, and P would be outside the rectangle. So $a^2 = \frac{26}{5}$, and the area is

$$3a^2 = \frac{78}{5} = 15.6 \text{ square units.}$$

8. Prove that in every triangle, the midpoint of the segment connecting the center of an excircle (a circle tangent to the side lines from outside) and the center of the inscribed circle lies on the circumscribed circle. (15 points)

   Solution: Let I be the center of the inscribed circle and I_a the center of the excircle opposite A. Both are on the bisector of the angle at A. Let this bisector meet the circumscribed circle again at M. We show that M is the midpoint of II_a. Let α, β, γ be the angles of the triangle.

   $MB = MI$: $\angle MBC = \angle MAC = \dfrac{\alpha}{2}$ (inscribed angles on the arc MC), so $\angle MBI = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$. The angle $\angle MIB$ is an exterior angle of triangle ABI, so it is $\dfrac{\alpha}{2} + \dfrac{\beta}{2}$ too. So triangle MBI is isosceles, and $MB = MI$.

   On the other hand, BI and BI_a are the interior and exterior bisectors at B, so $\angle IBI_a = 90^\circ$, and B is on the circle with diameter II_a. So the midpoint N of II_a satisfies $NB = NI$.

   Both M and N are on the line AI, and both are on the perpendicular bisector of BI. This perpendicular bisector meets the line AI in only one point, so $M = N$. So the midpoint of II_a is M, which is on the circumscribed circle. (The same argument works for the other two excircles.)

Recommended for students from vocational secondary schools

6. Prove that if the difference of the squares of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\dfrac{c^2}{a^2}$, then

$$b^4 - c^4 = 4ab^2c$$

   (13 points)

   Solution: The equation as printed in the original is $ax^2 + bx - c = 0$. With that equation, the same computation gives $b^4 - c^4 = -4ab^2c$, so the minus sign is evidently a misprint, and we prove the statement for $ax^2 + bx + c = 0$.

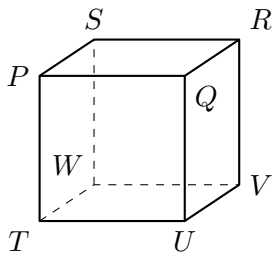
   Let the roots be x_1 and x_2. By Vieta's formulas, $x_1 + x_2 = -\dfrac{b}{a}$ and $x_1x_2 = \dfrac{c}{a}$. The condition is

$$x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) = -\frac{b}{a}(x_1 - x_2) = \frac{c^2}{a^2}, \quad \text{so} \quad x_1 - x_2 = -\frac{c^2}{ab}$$

   Squaring, and using $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$:

$$\frac{c^4}{a^2b^2} = \frac{b^2}{a^2} - \frac{4c}{a}, \qquad c^4 = b^4 - 4ab^2c, \qquad b^4 - c^4 = 4ab^2c$$

7. The vertices of a cube are labeled as in the figure. Let A be the center of face $TUVW$, and B the center of face $UVRQ$. The edge UV intersects the plane PAB at X. Find the value of the ratio $\dfrac{UX}{XV}$.



(14 points)

Solution: Place the cube in a coordinate system with edge 1: $T\,(0,0,0)$, $U\,(1,0,0)$, $V\,(1,1,0)$, $W\,(0,1,0)$, and P, Q, R, S above them, so $P\,(0,0,1)$. Then $A=\left(\frac{1}{2},\frac{1}{2},0\right)$ and $B=\left(1,\frac{1}{2},\frac{1}{2}\right)$. A normal vector of the plane PAB is

$$\overrightarrow{PA}\times\overrightarrow{PB}=\left(\frac{1}{2},\frac{1}{2},-1\right)\times\left(1,\frac{1}{2},-\frac{1}{2}\right)=\left(\frac{1}{4},-\frac{3}{4},-\frac{1}{4}\right)$$

so the plane is $x-3y-z+1=0$. The points of UV are $(1,t,0)$, and $1-3t+1=0$ gives $t=\frac{2}{3}$. So $UX=\frac{2}{3}$, $XV=\frac{1}{3}$, and $\frac{UX}{XV}=2$.

8. The real numbers x, y, z satisfy

$$x+y=z-1,\qquad xy=z^2-7z+10$$

For which values of x, y, z is the sum x^2+y^2 maximal? (15 points)

Solution: We express x^2+y^2 in terms of z:

$$x^2+y^2=(x+y)^2-2xy=(z-1)^2-2\left(z^2-7z+10\right)=-z^2+12z-19$$

But z cannot be arbitrary: x and y are real roots of $t^2-(z-1)\,t+\left(z^2-7z+10\right)=0$, so the discriminant is non-negative:

$$(z-1)^2-4\left(z^2-7z+10\right)\geq 0,\qquad 3z^2-26z+39\leq 0,\qquad \frac{13-2\sqrt{13}}{3}\leq z\leq\frac{13+2\sqrt{13}}{3}$$

that is, approximately $1.93\leq z\leq 6.74$. The function $-z^2+12z-19$ has its maximum at $z=6$, which is in this interval, and the maximum is 17. Then $x+y=5$ and $xy=4$, so the maximum $x^2+y^2=17$ is attained at $x=1$, $y=4$, $z=6$ and at $x=4$, $y=1$, $z=6$.

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