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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equation and inequalities.

$$\text{a) } \sqrt{4x + x^2} = 4 - x \quad \text{b) } \sqrt{4x + x^2} < 4 - x \quad \text{c) } \sqrt{4x + x^2} > 4 - x$$

Solution: The square root is defined if $x^2 + 4x = x(x + 4) \geq 0$, that is, $x \leq -4$ or $x \geq 0$. If $4 - x \geq 0$, that is, $x \leq 4$, then both sides are nonnegative, and squaring is allowed:

$$4x + x^2 \text{ vs. } 16 - 8x + x^2, \quad \text{that is,} \quad 12x \text{ vs. } 16, \quad x \text{ vs. } \frac{4}{3}$$

a) $x = \frac{4}{3}$ (it is in the domain and $x \leq 4$).

b) We need $x \leq 4$ (otherwise the right-hand side is negative) and $x < \frac{4}{3}$, together with the domain: $x \leq -4$ or $0 \leq x < \frac{4}{3}$.

c) For $x > 4$, the right-hand side is negative and the inequality is true. For $x \leq 4$, we need $x > \frac{4}{3}$. So the solution is $x > \frac{4}{3}$.

2. The parallel sides of a trapezoid are 1 and 4 units long, and its legs are 2 and 3 units long. Find the lengths of the diagonals of the trapezoid.

Solution: Let the longer base be $AB = 4$ with $A(0, 0)$ and $B(4, 0)$, and let the shorter base $DC = 1$ be at height h , with $AD = 2$ and $BC = 3$. The projections of the legs onto AB are p and q with $p + q = 4 - 1 = 3$, and

$$p^2 + h^2 = 4, \quad q^2 + h^2 = 9$$

So $q^2 - p^2 = 5$, $(q - p) \cdot 3 = 5$, $q - p = \frac{5}{3}$, so $p = \frac{2}{3}$, $q = \frac{7}{3}$, and $h^2 = 4 - \frac{4}{9} = \frac{32}{9}$. Then $D = \left(\frac{2}{3}, h\right)$ and $C = \left(\frac{5}{3}, h\right)$, and the diagonals are

$$AC = \sqrt{\frac{25}{9} + \frac{32}{9}} = \sqrt{\frac{57}{9}} \approx 2.52, \quad BD = \sqrt{\frac{100}{9} + \frac{32}{9}} = \sqrt{\frac{132}{9}} \approx 3.83$$

3. Solve the given system of equations.

$$\begin{aligned}x^2 - xy - 2y^2 &= 0 \\ 2x^2 + y^2 + \sqrt{2x^2 + y^2} &= 12\end{aligned}$$

Solution: Let $w = \sqrt{2x^2 + y^2} \geq 0$. The second equation is $w^2 + w - 12 = 0$, that is, $(w + 4)(w - 3) = 0$, so $w = 3$ and $2x^2 + y^2 = 9$. The first equation factors: $(x - 2y)(x + y) = 0$.

If $x = 2y$: $9y^2 = 9$, so $y = \pm 1$, and we get $(2, 1)$ and $(-2, -1)$.

If $x = -y$: $3y^2 = 9$, so $y = \pm\sqrt{3}$, and we get $(-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}, -\sqrt{3})$.

The system has four solutions: $(2, 1)$, $(-2, -1)$, $(\sqrt{3}, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$.

4. For what values of the parameter a are the roots of the equation

$$x^2 - 2ax + 2a^2 - 3a = 0$$

real, and for which of these a does $(x_1 - x_2)^2$ take its largest value, where x_1 and x_2 are the roots of the given equation?

Solution: The discriminant is

$$D = 4a^2 - 4(2a^2 - 3a) = 4(3a - a^2) = 4a(3 - a)$$

so the roots are real for $0 \leq a \leq 3$. Then $x_{1,2} = a \pm \frac{\sqrt{D}}{2}$, so

$$(x_1 - x_2)^2 = D = 12a - 4a^2 = 9 - 4\left(a - \frac{3}{2}\right)^2$$

which is largest for $a = \frac{3}{2}$ (which is in the interval $[0, 3]$), and its largest value is 9.

5. The vertex C of the square $ABCD$ is on the y -axis, and the opposite vertex is $A(2, 2)$. The area of the square is 20 square units. Find the coordinates of the unknown vertices of the square.

Solution: The diagonal of a square of area 20 is $\sqrt{40}$. Let $C = (0, c)$; then $AC^2 = 4 + (c - 2)^2 = 40$, so $c - 2 = \pm 6$, and $c = 8$ or $c = -4$. The vertices B and D are obtained from the center O of the square by adding the vector $\overrightarrow{OA}$ rotated by $\pm 90^\circ$ (rotating (u, v) by 90° gives $(-v, u)$).

If $C = (0, 8)$: $O = (1, 5)$, $\overrightarrow{OA} = (1, -3)$, rotated: $\pm(3, 1)$. So the vertices are $(4, 6)$ and $(-2, 4)$.

If $C = (0, -4)$: $O = (1, -1)$, $\overrightarrow{OA} = (1, 3)$, rotated: $\pm(-3, 1)$. So the vertices are $(-2, 0)$ and $(4, -2)$.

There are two such squares: B and D are $(4, 6)$ and $(-2, 4)$ with $C(0, 8)$, or $(4, -2)$ and $(-2, 0)$ with $C(0, -4)$ (which one is B and which is D depends on the orientation).

6. Solve the given equation.

$$3(\cot x - \tan x) = 2 \tan 2x$$

Solution: We need $\sin x \neq 0$, $\cos x \neq 0$, $\cos 2x \neq 0$. Since

$$\cot x - \tan x = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\frac{1}{2} \sin 2x} = 2 \cot 2x$$

the equation is $6 \cot 2x = 2 \tan 2x$, that is, $\tan^2 2x = 3$, $\tan 2x = \pm\sqrt{3}$. So $2x = \pm\frac{\pi}{3} + k\pi$, and

$$x = \pm\frac{\pi}{6} + \frac{k\pi}{2}, \quad k \text{ is an integer}$$

(These are allowed values.)

7. The first term of a geometric sequence is 3, the sum of the first n terms is 33, and the sum of the reciprocals of the first n terms is $\frac{11}{48}$. Find the first n terms of the sequence.

Solution: Let the ratio be q ; $q = 1$ is impossible ($3n = 33$ would give $n = 11$, but then the sum of the reciprocals would be $\frac{11}{3}$). The reciprocals $\frac{1}{3}, \frac{1}{3q}, \dots, \frac{1}{3q^{n-1}}$ are the terms of the original sequence in reverse order, divided by $9q^{n-1}$:

$$\frac{1}{3} + \frac{1}{3q} + \dots + \frac{1}{3q^{n-1}} = \frac{3q^{n-1} + 3q^{n-2} + \dots + 3}{9q^{n-1}} = \frac{33}{9q^{n-1}}$$

So $\frac{33}{9q^{n-1}} = \frac{11}{48}$, and $q^{n-1} = 16$: the n th term is $3 \cdot 16 = 48$. The sum of the terms is $\frac{48q - 3}{q - 1} = 33$, so $48q - 3 = 33q - 33$, and $q = -2$. Then $(-2)^{n-1} = 16$, $n = 5$. The terms are

$$3, -6, 12, -24, 48$$

(Check: the sum is 33, and $\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \frac{1}{24} + \frac{1}{48} = \frac{16 - 8 + 4 - 2 + 1}{48} = \frac{11}{48}$.)

8. A cube is inscribed in a regular square pyramid so that four vertices of the cube are on the base of the pyramid, and the other four vertices are on the lateral edges of the pyramid. Prove that

$$V_1 \leq \frac{4}{9}V$$

where V_1 is the volume of the cube, and V is the volume of the pyramid. When does equality hold?

Solution: Let the base edge of the pyramid be a , its height m , and the edge of the cube b . The upper vertices of the cube are on the lateral edges, so the upper face of the cube is the cross-section of the pyramid at height b , which is a square with side $a\left(1 - \frac{b}{m}\right)$. So

$$b = a\left(1 - \frac{b}{m}\right), \quad b = \frac{am}{a + m}$$

We have to prove that

$$\left(\frac{am}{a + m}\right)^3 \leq \frac{4}{9} \cdot \frac{a^2m}{3}, \quad \text{that is, } 27am^2 \leq 4(a + m)^3$$

(we divided by $a^2m > 0$ and multiplied by $27(a + m)^3 > 0$). Expanding,

$$4(a + m)^3 - 27am^2 = 4a^3 + 12a^2m - 15am^2 + 4m^3 = (a + 4m)(2a - m)^2 \geq 0$$

which is true. Equality holds if and only if $m = 2a$, that is, the height of the pyramid is twice its base edge (then $b = \frac{2a}{3}$).

(Another way: by the inequality between the arithmetic and geometric means of the three positive numbers $2a, m, m$: $\frac{2a + m + m}{3} \geq \sqrt[3]{2am^2}$, which gives the same inequality.)