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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation over the real numbers.

$$3^x \cdot 4^{\frac{x}{x+1}} = 6$$

Solution: We need $x \neq -1$. Taking the logarithm (any base) of both sides and multiplying by $x + 1$:

$$x(x+1)\log 3 + 2x\log 2 = (x+1)(\log 2 + \log 3), \quad x^2\log 3 + x\log 2 - (\log 2 + \log 3) = 0$$

$x = 1$ is a root (the coefficients add up to 0). The product of the roots is $-\frac{\log 2 + \log 3}{\log 3} = -\frac{\log 6}{\log 3}$, so the other root is $x = -\log_3 6 = -1 - \log_3 2 \approx -1.631$. The solutions are

$$x = 1 \quad \text{and} \quad x = -\log_3 6$$

2. Find the equation of the circle that passes through the points $A(0, 1)$ and $B(0, 3)$ and is tangent to the x -axis.

Solution: The center is on the perpendicular bisector of AB , which is the line $y = 2$. The circle is tangent to the x -axis, so its radius is 2, and the center $(h, 2)$ is 2 units away from A :

$$h^2 + 1 = 4, \quad h = \pm\sqrt{3}$$

There are two such circles:

$$(x - \sqrt{3})^2 + (y - 2)^2 = 4, \quad (x + \sqrt{3})^2 + (y - 2)^2 = 4$$

3. The product of the first and fifth terms of a geometric sequence is 144, and the difference of the fourth and second terms is 18. Find the first term and the ratio of the sequence.

Solution: Let the first term be a and the ratio q . Then $a_1 a_5 = a^2 q^4 = (aq^2)^2 = 144$, so the third term is $a_3 = aq^2 = \pm 12$. Also $a_4 - a_2 = a_3 q - \frac{a_3}{q} = 18$.

If $a_3 = 12$: $q - \frac{1}{q} = \frac{3}{2}$, $2q^2 - 3q - 2 = 0$, so $q = 2$ (and $a = 3$) or $q = -\frac{1}{2}$ (and $a = 48$).

If $a_3 = -12$: $q - \frac{1}{q} = -\frac{3}{2}$, $2q^2 + 3q - 2 = 0$, so $q = \frac{1}{2}$ (and $a = -48$) or $q = -2$ (and $a = -3$).

There are four such sequences:

$$\begin{aligned} a = 3, q = 2 : & \quad 3, 6, 12, 24, 48 \\ a = 48, q = -\frac{1}{2} : & \quad 48, -24, 12, -6, 3 \\ a = -48, q = \frac{1}{2} : & \quad -48, -24, -12, -6, -3 \\ a = -3, q = -2 : & \quad -3, 6, -12, 24, -48 \end{aligned}$$

4. Consider the right circular cones for which $b + 2r = 8$, where b is the slant height of the cone, and r is the radius of its base. Which of these cones has the largest lateral surface area? Find the volume of this cone.

Solution: The lateral surface area is

$$\pi r b = \pi r (8 - 2r) = 2\pi (4r - r^2) = 2\pi (4 - (r - 2)^2)$$

which is largest for $r = 2$, and then $b = 4$ (this is a cone, since $b > r$). The largest lateral surface area is 8π . The height of this cone is $m = \sqrt{b^2 - r^2} = \sqrt{12} = 2\sqrt{3}$, so its volume is

$$V = \frac{1}{3} r^2 \pi m = \frac{1}{3} \cdot 4\pi \cdot 2\sqrt{3} = \frac{8\sqrt{3}}{3} \pi \approx 14.51$$

5. The numbers p and q are real numbers such that the roots of the equations $4x^2 - 4px + q = 0$ and $4x^2 - 4p^2x - pq = 0$ are real, and the roots of the second equation are 1 greater than the roots of the first equation. Find the values of p and q .

Solution: Let the roots of the first equation be x_1 and x_2 ; then $x_1 + x_2 = p$ and $x_1 x_2 = \frac{q}{4}$. The roots of the second equation are $x_1 + 1$ and $x_2 + 1$, so

$$(x_1 + 1) + (x_2 + 1) = p^2, \quad (x_1 + 1)(x_2 + 1) = -\frac{pq}{4}$$

The first condition gives $p + 2 = p^2$, so $p = 2$ or $p = -1$. The second one gives

$$\frac{q}{4} + p + 1 = -\frac{pq}{4}, \quad (p + 1) \left(\frac{q}{4} + 1 \right) = 0$$

If $p = 2$: $q = -4$. Then the equations are $4x^2 - 8x - 4 = 0$ with roots $1 \pm \sqrt{2}$, and $4x^2 - 16x + 8 = 0$ with roots $2 \pm \sqrt{2}$, which is correct.

If $p = -1$: the second condition is true for every q . The equations are $4x^2 + 4x + q = 0$ and $4x^2 - 4x + q = 0$; the roots are real if $q \leq 1$, and then they are $\frac{-1 \pm \sqrt{1-q}}{2}$ and $\frac{1 \pm \sqrt{1-q}}{2}$, so the roots of the second equation are indeed 1 greater.

So $p = 2$ and $q = -4$, or $p = -1$ and q is any number with $q \leq 1$.

6. A convex hexagon can be inscribed in a circle of radius r so that three of its sides are 2 units long and the other three sides are 5 units long. Find the radius of the circle.

Solution: Let the central angle over a side of length 2 be α , and over a side of length 5 be β . The six central angles add up to 360° , so $3\alpha + 3\beta = 360^\circ$, that is, $\alpha + \beta = 120^\circ$. Somewhere around the hexagon, a side $PQ = 2$ is followed by a side $QR = 5$. The arc PR containing Q is then 120° , so the inscribed angle PQR is $180^\circ - 60^\circ = 120^\circ$. By the Law of Cosines,

$$PR^2 = 2^2 + 5^2 - 2 \cdot 2 \cdot 5 \cos 120^\circ = 39, \quad PR = \sqrt{39}$$

and by the Law of Sines in the triangle PQR (whose circumscribed circle is our circle),

$$2r = \frac{PR}{\sin 120^\circ} = \frac{\sqrt{39}}{\frac{\sqrt{3}}{2}} = 2\sqrt{13}, \quad r = \sqrt{13} \approx 3.61$$

7. For which real numbers p does the equation $\log_{1-x}(px) = 2$ have two different real solutions?

Solution: The base must be positive and not 1, so $x < 1$ and $x \neq 0$, and we need $px > 0$. The equation says $(1-x)^2 = px$, that is,

$$x^2 - (2+p)x + 1 = 0$$

(If x is a root with $x \neq 1$, then $px = (1-x)^2 > 0$ automatically.) We need two different roots, both less than 1 (and not 0, but 0 is never a root). The product of the roots is 1, so they have the same sign. If both were positive, then one of them would be greater than 1 (two different positive numbers with product 1). So both roots must be negative: their sum $2+p$ is negative, and the discriminant is positive:

$$p < -2, \quad (2+p)^2 - 4 = p(p+4) > 0$$

So the equation has two different solutions exactly when $p < -4$.

8. Find all pairs of numbers (x, y) that satisfy the given equation.

$$4 \sin^2 x - 4 \cos x \sin y - 5 = 0$$

Solution: Using $\sin^2 x = 1 - \cos^2 x$ and multiplying by -1 :

$$4 \cos^2 x + 4 \cos x \sin y + 1 = 0, \quad (2 \cos x + \sin y)^2 + 1 - \sin^2 y = 0, \quad (2 \cos x + \sin y)^2 + \cos^2 y = 0$$

So $\cos y = 0$ and $2 \cos x = -\sin y$.

If $\sin y = 1$, that is, $y = \frac{\pi}{2} + 2k\pi$: $\cos x = -\frac{1}{2}$, $x = \pm \frac{2\pi}{3} + 2n\pi$.

If $\sin y = -1$, that is, $y = -\frac{\pi}{2} + 2k\pi$: $\cos x = \frac{1}{2}$, $x = \pm \frac{\pi}{3} + 2n\pi$.

Here k and n are arbitrary integers.