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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The first digit of a four-digit number is 4. If we delete its last digit and write this digit in front of the remaining three digits, we get a number that is 2826 greater. What is this four-digit number?

Solution: Let the number be $\overline{4yc}$, where y is the two-digit block in the middle ($0 \leq y \leq 99$) and c is the last digit. So the number is $4000 + 10y + c$, and the new number is $1000c + 400 + y$. Then

$$1000c + 400 + y = 4000 + 10y + c + 2826, \quad 999c - 9y = 6426, \quad y = 111c - 714$$

Since $0 \leq y \leq 99$, we need $714 \leq 111c \leq 813$, so $c = 7$ and $y = 63$. The number is 4637. (Check: $7463 - 4637 = 2826$.)

2. In a triangle, $\tan \frac{\alpha}{2} = \frac{5}{6}$ and $\tan \frac{\beta}{2} = \frac{20}{37}$. Find $\tan \frac{\gamma}{2}$. Show that $b = \frac{a+c}{2}$.

Solution: Since $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$,

$$\tan \frac{\gamma}{2} = \cot \left(\frac{\alpha}{2} + \frac{\beta}{2} \right) = \frac{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}} = \frac{1 - \frac{100}{222}}{\frac{5}{6} + \frac{20}{37}} = \frac{\frac{122}{222}}{\frac{305}{222}} = \frac{2}{5}$$

Let r be the radius of the inscribed circle and s the semiperimeter. The tangent segments from A to the inscribed circle have length $s - a$, so $\tan \frac{\alpha}{2} = \frac{r}{s - a}$, and similarly for the other angles. So

$$s - a = \frac{6}{5}r = \frac{24}{20}r, \quad s - b = \frac{37}{20}r, \quad s - c = \frac{5}{2}r = \frac{50}{20}r$$

Adding these, $s = (s - a) + (s - b) + (s - c) = \frac{111}{20}r$, and so

$$a = \frac{87}{20}r, \quad b = \frac{74}{20}r, \quad c = \frac{61}{20}r$$

Indeed, $a + c = \frac{148}{20}r = 2b$.

3. A ship leaves P for Q , going upstream on a river, and at the same time a boat leaves Q for P . They meet when the boat has covered one third of its way. The ship waits 1 hour in Q , then turns back, and arrives in P 1 hour before the boat. If the boat doubled its speed (in still water), then it would arrive in P 3 hours before the ship. How long did it take for the ship to get from P back to P ?

Solution: Let d be the distance PQ , w the speed of the current, and u and v the speeds of the ship and the boat in still water. Let T be the time from the ship's departure until its return to P . The boat goes downstream with speed $v + w$, and its trip takes $T + 1$ hours. Until they meet, the ship covers twice the distance the boat covers, so the ship's upstream speed is twice the boat's speed: the ship's upstream trip takes $\frac{T + 1}{2}$ hours. Then the downstream trip of the ship takes $T - 1 - \frac{T + 1}{2} = \frac{T - 3}{2}$ hours. With doubled speed, the boat's trip would take $T - 3$ hours. So

$$v + w = \frac{d}{T + 1} \quad (\text{the boat})$$

$$u - w = \frac{2d}{T + 1} \quad (\text{the ship upstream})$$

$$u + w = \frac{2d}{T - 3} \quad (\text{the ship downstream})$$

$$2v + w = \frac{d}{T - 3} \quad (\text{the faster boat})$$

From the second and third equations, $2w = \frac{2d}{T - 3} - \frac{2d}{T + 1}$; from the first and fourth ones, $w = 2(v + w) - (2v + w) = \frac{2d}{T + 1} - \frac{d}{T - 3}$. So

$$\frac{d}{T - 3} - \frac{d}{T + 1} = \frac{2d}{T + 1} - \frac{d}{T - 3}, \quad \frac{2}{T - 3} = \frac{3}{T + 1}, \quad T = 11$$

The ship got back to P in 11 hours (including the 1 hour of waiting): it went 6 hours upstream and 4 hours downstream. (With $d = 24$, the speeds are $w = 1$, $u = 5$, $v = 1$, and everything checks.)

4. Solve the given equation over the real numbers.

$$(x^2 - 1988^2)^2 - 7952x - 1 = 0$$

Solution: Let $a = 1988$; then $7952 = 4a$, and the equation is $(x^2 - a^2)^2 = 4ax + 1$. Adding $4a^2x^2$ to both sides, the left-hand side becomes

$$(x^2 - a^2)^2 + 4a^2x^2 = (x^2 + a^2)^2$$

and the right-hand side becomes $4a^2x^2 + 4ax + 1 = (2ax + 1)^2$. So $x^2 + a^2 = \pm(2ax + 1)$.

With the $+$ sign: $x^2 - 2ax + a^2 = 1$, $(x - a)^2 = 1$, $x = a \pm 1$.

With the $-$ sign: $x^2 + 2ax + a^2 = -1$, $(x + a)^2 = -1$, which has no real solution.

The solutions are $x = 1987$ and $x = 1989$.

5. The elements of a set C are natural numbers. The least common multiple of the elements is 2210. No two elements of the set are relatively prime. The product of the elements is not divisible by 416. At most how many elements can C have, and what are these numbers?

Solution: $2210 = 2 \cdot 5 \cdot 13 \cdot 17$ and $416 = 2^5 \cdot 13$. Every element divides 2210, and 1 is not an element (it is relatively prime to everything). The 16 divisors of 2210 form 8 pairs $\left\{d, \frac{2210}{d}\right\}$:

$$\{1, 2210\}, \{2, 1105\}, \{5, 442\}, \{10, 221\}, \{13, 170\}, \{26, 85\}, \{17, 130\}, \{34, 65\}$$

The two numbers in a pair are relatively prime, so C contains at most one number from each pair, and C has at most 8 elements.

Some element is divisible by 13 (because of the least common multiple), so the product is not divisible by 416 exactly when at most 4 elements are even. If C has 8 elements, it has exactly one number from each pair, and since one number in each pair is even, at least 4 of the elements are odd. The odd elements are products of some of 5, 13, 17, and no two of them are relatively prime. There are only 4 ways to choose 4 such odd numbers (more than 4 is impossible, since the 7 odd divisors other than 1 form three pairs $\{5, 221\}$, $\{13, 85\}$, $\{17, 65\}$ and 1105):

$$\{5, 65, 85, 1105\}, \{13, 65, 221, 1105\}, \{17, 85, 221, 1105\}, \{65, 85, 221, 1105\}$$

The other four elements are then the even numbers from the remaining pairs, and in each case we can check that no two elements are relatively prime. So C has at most 8 elements, and there are four such sets:

$$\begin{aligned} &\{5, 10, 65, 85, 130, 170, 1105, 2210\} \\ &\{13, 26, 65, 130, 221, 442, 1105, 2210\} \\ &\{17, 34, 85, 170, 221, 442, 1105, 2210\} \\ &\{65, 85, 130, 170, 221, 442, 1105, 2210\} \end{aligned}$$

6. For what values of the parameter k does the quadratic equation

$$(k+10)x^2 - 2(k+10)(k-5)x + (k-5)(k-18) = 0$$

have two different roots with the same sign?

Solution: The equation is quadratic if $k \neq -10$. We need a positive discriminant:

$$\begin{aligned} \frac{D}{4} &= (k+10)^2(k-5)^2 - (k+10)(k-5)(k-18) = (k+10)(k-5)(k^2+4k-32) \\ &= (k+10)(k+8)(k-4)(k-5) > 0 \end{aligned}$$

which holds for $k < -10$, for $-8 < k < 4$, and for $k > 5$. The roots have the same sign (and are not zero) if their product is positive:

$$x_1x_2 = \frac{(k-5)(k-18)}{k+10} > 0$$

which holds for $-10 < k < 5$ and for $k > 18$. Both conditions hold for

$$-8 < k < 4 \quad \text{or} \quad k > 18$$

7. Out of 66 actors, any two have already played together either in a theater performance, or in a film, or in a TV play, or in dubbing, but each pair played together in only one kind of production. Prove that there are three actors among them such that any two of them played together in the same kind of production.

Solution: Pick an actor A . The other 65 actors are in 4 groups according to the kind of production they played in with A . Since $65 > 4 \cdot 16$, one group has at least 17 actors; say these 17 played with A in a theater performance. If any two of them also played together in a theater performance, then these two and A are three such actors. Otherwise any two of these 17 actors played together in one of the other 3 kinds of production.

Pick one of the 17, say B . The other 16 are in 3 groups according to the kind of production with B , and since $16 > 3 \cdot 5$, one group has at least 6 actors; say they played with B in a film. If any two of them played together in a film, we are done. Otherwise any two of these 6 actors played together in a TV play or in dubbing.

Pick one of the 6, say C . Of the other 5, at least 3 played with C in the same kind of production, say in a TV play. If two of these 3 also played together in a TV play, then they and C are three such actors. If not, then all three of them played together in dubbing, and they are three such actors.

8. Prove that in every triangle,

$$r_a + r_b + r_c - r_0 = 4R$$

where r_a, r_b, r_c are the radii of the excircles tangent to the sides a, b, c , r_0 is the radius of the inscribed circle, and R is the radius of the circumscribed circle.

Solution: Let T be the area and s the semiperimeter. Then $r_0 = \frac{T}{s}$, $r_a = \frac{T}{s-a}$, $r_b = \frac{T}{s-b}$, $r_c = \frac{T}{s-c}$, and $T = \frac{abc}{4R}$. Now

$$r_a + r_b = T \cdot \frac{(s-b) + (s-a)}{(s-a)(s-b)} = \frac{Tc}{(s-a)(s-b)}, \quad r_c - r_0 = T \cdot \frac{s - (s-c)}{s(s-c)} = \frac{Tc}{s(s-c)}$$

so

$$r_a + r_b + r_c - r_0 = Tc \cdot \frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$

By Heron's formula, the denominator is T^2 . The numerator is

$$s^2 - sc + s^2 - s(a+b) + ab = 2s^2 - s(a+b+c) + ab = 2s^2 - 2s^2 + ab = ab$$

So

$$r_a + r_b + r_c - r_0 = \frac{Tc \cdot ab}{T^2} = \frac{abc}{T} = 4R$$