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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the smallest value of the function defined on the real numbers by

$$f(x) = \frac{(x^2 - 8x + 8)^2 - 100}{x^2 - 8x + 18}$$

Solution: Let $u = x^2 - 8x + 18 = (x - 4)^2 + 2$; then $u \geq 2$ (so the function is defined everywhere), and $x^2 - 8x + 8 = u - 10$. So

$$f(x) = \frac{(u - 10)^2 - 100}{u} = \frac{u^2 - 20u}{u} = u - 20$$

which is smallest when u is smallest, that is, $u = 2$ at $x = 4$. The smallest value of the function is $f(4) = -18$.

2. In triangle ABC , $AB = 13$, $BC = 14$, $CA = 15$. Let M be the midpoint of the side AB , and T the foot of the altitude to the side BC . Find the length of the segment MT .

Solution: T is the foot of the altitude from A , so the triangle ABT has a right angle at T . (The angles of triangle ABC at B and C are acute, since $15^2 < 13^2 + 14^2$ and $13^2 < 14^2 + 15^2$, so T is on the segment BC ; but this is not even needed.) By Thales' theorem, T is on the circle with diameter AB , whose center is M . So

$$MT = \frac{AB}{2} = 6.5$$

3. Two vehicles move back and forth between the places A and B , with speeds v_1 and v_2 . They start at the same time from A and from B , towards each other. After how much time do they meet exactly at the midpoint of the road for the second time, if $AB = 150$ km, $v_1 = 50$ km/h, and $v_2 = 30$ km/h?

Solution: The first vehicle covers the road AB in 3 hours, so it is at the midpoint at the times

$$1.5, 4.5, 7.5, 10.5, \dots, \text{ that is, } 1.5 + 3k \text{ hours}$$

The second vehicle covers the road in 5 hours, so it is at the midpoint at the times

$$2.5, 7.5, 12.5, \dots, \text{ that is, } 2.5 + 5n \text{ hours}$$

They meet at the midpoint when $1.5 + 3k = 2.5 + 5n$, that is, $3k - 5n = 1$ for nonnegative integers k and n . The solutions are $k = 2 + 5j$, $n = 1 + 3j$, so the meetings at the midpoint are at 7.5, 22.5, 37.5, ... hours. The second one is after 22.5 hours.

(They meet at other places too: the meetings before that are after 1.875, 5.625 (when the first vehicle overtakes the second), 7.5, 9.375, 13.125, 16.875, and 20.625 hours.)

4. In triangle ABC , the angle CBA is 72° . Let F be the midpoint of the side AC , and let D be the point dividing the side BC into thirds that is closer to B . The segments AD and BF intersect at P . Find the ratio of the areas of the triangle BDP and the quadrilateral $PDCF$.

Solution: Let T be the area of triangle ABC . The line through F parallel to BC intersects AD at its midpoint G , and $FG = \frac{DC}{2} = \frac{BC}{3} = BD$ (midline in triangle ADC). So $BDFG$ is a parallelogram, its diagonals BF and DG bisect each other, and P is the midpoint of BF . The triangle BFC has area $\frac{T}{2}$ (since F is the midpoint of AC). The triangle BDP has base $BD = \frac{1}{3}BC$ and height half the height of BFC from F (since P is the midpoint of BF), so its area is

$$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{T}{2} = \frac{T}{12}$$

and the area of $PDCF$ is $\frac{T}{2} - \frac{T}{12} = \frac{5T}{12}$. The ratio is

$$T_{BDP} : T_{PDCF} = 1 : 5$$

(The angle 72° is not needed; the ratio is the same in every triangle.)

5. The hypotenuse of a right triangle is 10 cm, and the bisector of the right angle is $\frac{24\sqrt{2}}{7}$ cm long. Find the perimeter of the triangle.

Solution: Let the legs be a and b , and the bisector f . The bisector divides the triangle into two triangles with angles 45° at the right angle, so by adding their areas,

$$\frac{1}{2}af \sin 45^\circ + \frac{1}{2}bf \sin 45^\circ = \frac{ab}{2}, \quad f = \frac{\sqrt{2}ab}{a+b}$$

So $\frac{ab}{a+b} = \frac{24}{7}$. Let $s = a + b$; then $ab = \frac{24}{7}s$, and $a^2 + b^2 = 100$ gives

$$s^2 - 2 \cdot \frac{24}{7}s = 100, \quad 7s^2 - 48s - 700 = 0, \quad s = \frac{48 \pm 148}{14}$$

so $s = 14$ (the other root is negative). The perimeter is $14 + 10 = 24$ cm. (The legs are 6 cm and 8 cm.)

6. Solve the given equation over the real numbers.

$$\sin 3x \cdot \sin 5x + 2 \sin 5x - 2 \sin 3x - 3 = 0$$

Solution: The equation can be written as

$$(\sin 3x + 2)(\sin 5x - 2) = -1$$

Here $1 \leq \sin 3x + 2 \leq 3$ and $-3 \leq \sin 5x - 2 \leq -1$, so the product is at most -1 , and it is equal to -1 only if $\sin 3x = -1$ and $\sin 5x = 1$. So

$$x = -\frac{\pi}{6} + \frac{2k\pi}{3}, \quad x = \frac{\pi}{10} + \frac{2n\pi}{5}$$

for some integers k and n . Multiplying by $\frac{30}{\pi}$: $-5 + 20k = 3 + 12n$, that is, $5k - 3n = 2$, so $k = 1 + 3j$ and $n = 1 + 5j$. The solutions are

$$x = \frac{\pi}{2} + 2j\pi, \quad j \text{ is an integer}$$

7. A line through the vertex A of a parabola is rotated around A ; its second intersection point with the parabola is M . Find the set of the points P on the chords AM for which

$$AP^2 = AM \cdot PM$$

Solution: Let $AP = \lambda \cdot AM$ with $0 < \lambda < 1$; then $PM = (1 - \lambda)AM$, and the condition is $\lambda^2 = 1 - \lambda$, so $\lambda = \frac{\sqrt{5}-1}{2} \approx 0.618$. (So P divides AM in the golden ratio.) So P is the image of M under the dilation with center A and ratio λ , and the set of points P is the image of the parabola under this dilation, without the point A .

In coordinates: if A is the origin and the parabola is $y = ax^2$, then $M = (m, am^2)$ with $m \neq 0$, and $P = (\lambda m, \lambda am^2)$ satisfies $y = \frac{a}{\lambda}x^2$. The set of points P is the parabola

$$y = \frac{a}{\lambda}x^2 = \frac{\sqrt{5}+1}{2}ax^2$$

without its vertex A . (Every point of this parabola other than A is obtained, from the point M with $m = \frac{x}{\lambda}$.)

8. Find all pairs of integers (x, y) that satisfy the given equation.

$$xy + \frac{x}{y} + 2x = 75$$

Solution: We need $y \neq 0$. Multiplying by y :

$$x(y^2 + 2y + 1) = 75y, \quad x(y+1)^2 = 75y$$

Since y and $y+1$ are relatively prime, $(y+1)^2$ must divide $75 = 3 \cdot 5^2$. So $(y+1)^2 = 1$ or $(y+1)^2 = 25$, that is, $y = 0$ (not allowed), $y = -2$, $y = 4$, or $y = -6$.

$$y = -2 : \quad x = -150$$

$$y = 4 : \quad 25x = 300, \quad x = 12$$

$$y = -6 : \quad 25x = -450, \quad x = -18$$

The solutions are $(x, y) = (-150, -2)$, $(12, 4)$, and $(-18, -6)$.