

author: Imre Rábai

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The sides of triangle ABC are $AB = 13$, $BC = 17$, and $AC = 24$ units long. Let D be the point on the side AC for which $AD = 16$ units. Find the length of the segment BD .

Solution: By the Law of Cosines in triangle ABC ,

$$\cos \alpha = \frac{13^2 + 24^2 - 17^2}{2 \cdot 13 \cdot 24} = \frac{456}{624} = \frac{19}{26}$$

and then in triangle ABD ,

$$BD^2 = 13^2 + 16^2 - 2 \cdot 13 \cdot 16 \cdot \frac{19}{26} = 169 + 256 - 304 = 121, \quad BD = 11$$

2. Solve the given equation.

$$|x^2 - 2x - 1| = \frac{5x + 1}{3}$$

Solution: The left-hand side is nonnegative, so we need $5x + 1 \geq 0$, that is, $x \geq -\frac{1}{5}$.

If $x^2 - 2x - 1 = \frac{5x + 1}{3}$: $3x^2 - 11x - 4 = 0$, $(3x + 1)(x - 4) = 0$. Here $x = -\frac{1}{3} < -\frac{1}{5}$ is not allowed, and $x = 4$

works: $|16 - 8 - 1| = 7 = \frac{21}{3}$.

If $x^2 - 2x - 1 = -\frac{5x + 1}{3}$: $3x^2 - x - 2 = 0$, $(3x + 2)(x - 1) = 0$. Here $x = -\frac{2}{3}$ is not allowed, and $x = 1$ works:

$$|1 - 2 - 1| = 2 = \frac{6}{3}.$$

The solutions are $x = 1$ and $x = 4$.

3. Find the exact value of $\sin 3x$ if $\cos 2x = -\frac{1}{4}$.

Solution: $\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{5}{8}$, so $\sin x = \pm\sqrt{\frac{5}{8}} = \pm\frac{\sqrt{10}}{4}$. Then

$$\sin 3x = 3 \sin x - 4 \sin^3 x = \sin x \left(3 - 4 \sin^2 x \right) = \sin x \left(3 - \frac{5}{2} \right) = \frac{\sin x}{2} = \pm \frac{\sqrt{10}}{8}$$

Both signs occur (for x and $-x$), so $\sin 3x = \frac{\sqrt{10}}{8}$ or $\sin 3x = -\frac{\sqrt{10}}{8}$.

4. Solve the given inequality.

$$\log_{\frac{1}{2}} (5^{x+1} - 25^x) \geq -2$$

Solution: Since $-2 = \log_{\frac{1}{2}} 4$ and the function $\log_{\frac{1}{2}}$ is decreasing, the inequality means

$$0 < 5^{x+1} - 25^x \leq 4$$

Let $u = 5^x > 0$; then $5^{x+1} - 25^x = 5u - u^2$. The condition $5u - u^2 > 0$ gives $0 < u < 5$, and $5u - u^2 \leq 4$ gives $u^2 - 5u + 4 = (u - 1)(u - 4) \geq 0$, that is, $u \leq 1$ or $u \geq 4$. So $0 < u \leq 1$ or $4 \leq u < 5$, and the solution is

$$x \leq 0 \quad \text{or} \quad \log_5 4 \leq x < 1$$

5. The sum of the first, third, and fifth terms of a geometric sequence is 63. The difference of the fourth and second terms is 9 times the ratio. Write down the first five terms of the sequence.

Solution: Let the first term be a and the ratio $q \neq 0$. Then

$$a(1 + q^2 + q^4) = 63, \quad aq^3 - aq = 9q$$

The second equation gives $a(q^2 - 1) = 9$. Dividing the first equation by this one:

$$\frac{1 + q^2 + q^4}{q^2 - 1} = 7, \quad q^4 - 6q^2 + 8 = 0, \quad q^2 = 2 \text{ or } q^2 = 4$$

If $q^2 = 4$, then $a = 3$; if $q^2 = 2$, then $a = 9$. There are four such sequences:

$$\begin{array}{ll} a = 3, q = 2 : & 3, 6, 12, 24, 48 \\ a = 3, q = -2 : & 3, -6, 12, -24, 48 \\ a = 9, q = \sqrt{2} : & 9, 9\sqrt{2}, 18, 18\sqrt{2}, 36 \\ a = 9, q = -\sqrt{2} : & 9, -9\sqrt{2}, 18, -18\sqrt{2}, 36 \end{array}$$

6. Let $x > 0$, $y > 0$, and $x + y = 2$. Prove that

$$\left(2 + \frac{1}{x}\right) \left(2 + \frac{1}{y}\right) \geq 9$$

Solution: Expanding, and using $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{2}{xy}$:

$$\left(2 + \frac{1}{x}\right) \left(2 + \frac{1}{y}\right) = 4 + 2 \left(\frac{1}{x} + \frac{1}{y}\right) + \frac{1}{xy} = 4 + \frac{5}{xy}$$

By the inequality between the arithmetic and geometric means, $\sqrt{xy} \leq \frac{x+y}{2} = 1$, so $xy \leq 1$ and $\frac{5}{xy} \geq 5$. So the product is at least 9, with equality for $x = y = 1$.

7. The angle at vertex C of the acute triangle ABC is 45° . Prove that the distance of the orthocenter of the triangle from the vertex C is equal to the side AB .

Solution: Let H be the orthocenter, and let A' be the foot of the altitude from A on the side BC (it is on the side, since the triangle is acute). In the right triangle $AA'C$, the angle at C is 45° , so this triangle is isosceles: $AA' = CA'$.

Compare the right triangles $CA'H$ and $AA'B$ (both have a right angle at A'). The line CH is perpendicular to AB , so $\angle A'CH = 90^\circ - \beta$; and in the right triangle $AA'B$, $\angle A'AB = 90^\circ - \beta$. So the two right triangles have equal angles at C and at A , and equal legs $CA' = AA'$ next to these angles. So they are congruent, and their hypotenuses are equal: $CH = AB$.

(Another way: $CH = 2R \cos \gamma$ and $AB = 2R \sin \gamma$, where R is the radius of the circumscribed circle, and $\cos 45^\circ = \sin 45^\circ$.)

8. Two adjacent vertices of the parallelogram $ABCD$ are $A(8, 9)$ and $B(0, 3)$. A third vertex of the parallelogram is on the line $x + 2y = 6$. The area of the parallelogram $ABCD$ is 80 square units. Find the coordinates of the vertices C and D .

Solution: Let $\mathbf{v} = (v_1, v_2) = \overrightarrow{BC} = \overrightarrow{AD}$, so $C = (v_1, 3 + v_2)$ and $D = (8 + v_1, 9 + v_2)$. With $\overrightarrow{BA} = (8, 6)$, the area of the parallelogram is $|8v_2 - 6v_1| = 80$, that is, $3v_1 - 4v_2 = \pm 40$. (Equivalently: $AB = 10$, so the distance of the line CD from the line AB is 8.)

If C is on the line: $v_1 + 2(3 + v_2) = 6$, so $v_1 = -2v_2$, and $-10v_2 = \pm 40$, $v_2 = \mp 4$. This gives $C(8, -1)$, $D(16, 5)$, or $C(-8, 7)$, $D(0, 13)$.

If D is on the line: $(8 + v_1) + 2(9 + v_2) = 6$, so $v_1 = -20 - 2v_2$, and $-60 - 10v_2 = \pm 40$, $v_2 = -2$ or $v_2 = -10$. This gives $C(-16, 1)$, $D(-8, 7)$, or $C(0, -7)$, $D(8, -1)$.

There are four such parallelograms:

$$\begin{array}{ll} C(8, -1), & D(16, 5) \\ C(-8, 7), & D(0, 13) \\ C(-16, 1), & D(-8, 7) \\ C(0, -7), & D(8, -1) \end{array}$$