translated from: KoMaL, October 1988, pages 301-302
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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1988. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. Solve the given equation over the real numbers.

$$\sqrt{1+x\sqrt{x^2-24}}=x-1$$

(9 points)

Solution: We need $x \geq 1$ (the right-hand side is nonnegative) and $x^2 \geq 24$, so $x \geq \sqrt{24}$. Squaring,

$$1+x\sqrt{x^2-24}=x^2-2x+1, \qquad x\sqrt{x^2-24}=x\,(x-2)\,, \qquad \sqrt{x^2-24}=x-2$$

(we divided by $x>0$). Squaring again, $x^2-24=x^2-4x+4$, so $x=7$. Check: $\sqrt{1+7\sqrt{25}}=\sqrt{36}=6=7-1$. The solution is $x=7$.

2. The sum of the first three terms of an arithmetic sequence is 105. If we leave the first two numbers unchanged and add 180 to the third number, then we get the first three terms of a geometric sequence. Find the first three terms of the arithmetic sequence. (11 points)

Solution: Let the terms be $35-d$, 35, $35+d$ (their sum is 105). Then $35-d$, 35, $215+d$ is a geometric sequence, so the square of the middle term is the product of the other two:

$$(35-d)\,(215+d)=1225, \qquad d^2+180d-6300=0, \qquad (d-30)\,(d+210)=0$$

If $d=30$, the arithmetic sequence is 5, 35, 65 (and 5, 35, 245 is geometric with ratio 7). If $d=-210$, the arithmetic sequence is 245, 35, -175 (and 245, 35, 5 is geometric with ratio $\frac{1}{7}$).

Note: In KoMaL, this problem says to add 108 to the third number. That is most likely a misprint for 180: with 108, the equation becomes $d^2+108d-3780=0$, whose solutions $d=-54\pm 6\sqrt{186}$ are not nice numbers.

3. Which real numbers satisfy the given equation?

$$\log\left(4^{x-2}+9\right)-\log\left(2^{x-2}+1\right)+\log 2-1=0$$

(12 points)

Solution: Both arguments of the logarithm are positive for every x. Since $1=\log 10$, the equation says

$$\log\left(2\left(4^{x-2}+9\right)\right)=\log\left(10\left(2^{x-2}+1\right)\right), \qquad 2\left(4^{x-2}+9\right)=10\left(2^{x-2}+1\right)$$

Let $u=2^{x-2}>0$; then $4^{x-2}=u^2$, and

$$2u^2+18=10u+10, \qquad u^2-5u+4=0, \qquad (u-1)\,(u-4)=0$$

$u=1$ gives $x-2=0$, $x=2$; $u=4$ gives $x-2=2$, $x=4$. The solutions are $x=2$ and $x=4$.

4. One angle of a triangle is equal to the sum of the other two angles. The radius of the circle circumscribed around the triangle is 6 units, and the area of the square drawn on the shortest side is equal to the area of the triangle. Find the area of the triangle. (12 points)

   Solution: If $\gamma = \alpha + \beta$, then $2\gamma = 180°$, so the triangle has a right angle. The hypotenuse is a diameter of the circumscribed circle, so it is 12 units long. Let the legs be $a \leq b$; the shortest side is a. Then

$$a^2 = \frac{ab}{2}, \qquad b = 2a$$

   and by the Pythagorean theorem $a^2 + 4a^2 = 144$, so $a^2 = \frac{144}{5}$. The area of the triangle is

$$a^2 = \frac{144}{5} = 28.8 \text{ square units}$$

5. In triangle ABC, $AB = BC$. Let F be the midpoint of BC, and let T be the foot of the altitude to the side BC. Find the coordinates of the vertices B and C, if $A\,(-2, 1)$, $T\,(16, 1)$, and the second coordinate of F is 13. (14 points)

   Solution: The altitude AT is horizontal, so the line BC is vertical: $x = 16$. Let $B = (16, b)$; since $F = (16, 13)$ is the midpoint of BC, $C = (16, 26 - b)$. Then $BC = |2b - 26|$, and $AB = BC$ gives

$$18^2 + (b-1)^2 = (2b-26)^2, \qquad 3b^2 - 102b + 351 = 0, \qquad b^2 - 34b + 117 = 0$$

   so $b = 17 \pm \sqrt{172} = 17 \pm 2\sqrt{43}$. There are two triangles:

$$\begin{aligned} B\left(16, 17 + 2\sqrt{43}\right) &\approx (16, 30.11), \qquad C\left(16, 9 - 2\sqrt{43}\right) \approx (16, -4.11) \\ B\left(16, 17 - 2\sqrt{43}\right) &\approx (16, 3.89), \qquad C\left(16, 9 + 2\sqrt{43}\right) \approx (16, 22.11) \end{aligned}$$

   In the first one, T is on the side BC; in the second one, the angle at B is obtuse and T is on the extension of BC. (In both, $AB = BC = 4\sqrt{43} \pm 8$.)

Recommended for students from academic high schools

6. Let x, y, z be real numbers such that $xyz = 1$ and $1 + z + zx \neq 0$. Prove that the expression

$$\frac{1}{1+x+xy} + \frac{1}{1+y+yz} + \frac{1}{1+z+zx}$$

   is defined, and its value does not depend on the choice of x, y, z. (12 points)

   Solution: Since $xyz = 1$, none of x, y, z is zero. Using $xyz = 1$,

$$z\,(1+x+xy) = z + zx + xyz = 1 + z + zx, \qquad zx\,(1+y+yz) = zx + xyz + xyz \cdot z = 1 + z + zx$$

   So $1 + x + xy = \frac{1+z+zx}{z} \neq 0$ and $1 + y + yz = \frac{1+z+zx}{zx} \neq 0$, and the expression is defined. Its value is

$$\frac{z}{1+z+zx} + \frac{zx}{1+z+zx} + \frac{1}{1+z+zx} = \frac{1+z+zx}{1+z+zx} = 1$$

7. The tangents of the angles of a triangle are in the ratio $1 : 2 : 3$. What is the ratio of the sides of the triangle? (15 points)

   Solution: Let $\tan\alpha = t$, $\tan\beta = 2t$, $\tan\gamma = 3t$. Since $\gamma = 180^\circ - (\alpha + \beta)$,

   $$\tan\gamma = -\tan(\alpha+\beta) = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \qquad 3t = -\frac{3t}{1-2t^2}$$

   $t \neq 0$, so $1 - 2t^2 = -1$, $t^2 = 1$. If $t = -1$, all three tangents would be negative, that is, all three angles would be obtuse, which is impossible. So $\tan\alpha = 1$, $\tan\beta = 2$, $\tan\gamma = 3$, all three angles are acute, and

   $$\sin\alpha = \frac{1}{\sqrt{2}}, \qquad \sin\beta = \frac{2}{\sqrt{5}}, \qquad \sin\gamma = \frac{3}{\sqrt{10}}$$

   (using $\sin\varphi = \dfrac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}$ for acute angles). By the Law of Sines, the sides are in the ratio of the sines of the opposite angles; multiplying by $\sqrt{10}$:

   $$a : b : c = \frac{1}{\sqrt{2}} : \frac{2}{\sqrt{5}} : \frac{3}{\sqrt{10}} = \sqrt{5} : 2\sqrt{2} : 3$$

8. Let P be an arbitrary point in the plane of the quadrilateral $ABCD$. The reflection of P through A is Q, and the reflection of P through B is R. Prove that the quadrilateral $ABCD$ is a parallelogram if and only if the reflection of Q through D is the same point as the reflection of R through C. (15 points)

   Solution: Use position vectors (vectors from a fixed origin), denoted by the lowercase letters. The reflection of a point X through a point M is $2\mathbf{m} - \mathbf{x}$ (because M is the midpoint). So

   $$\mathbf{q} = 2\mathbf{a} - \mathbf{p}, \qquad \mathbf{r} = 2\mathbf{b} - \mathbf{p}$$

   The reflection of Q through D is $2\mathbf{d} - \mathbf{q} = 2\mathbf{d} - 2\mathbf{a} + \mathbf{p}$, and the reflection of R through C is $2\mathbf{c} - \mathbf{r} = 2\mathbf{c} - 2\mathbf{b} + \mathbf{p}$. These two points are the same if and only if

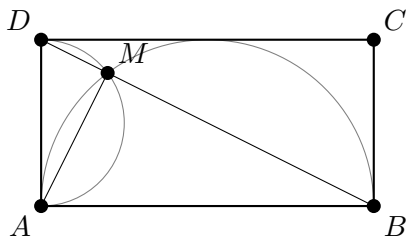
   $$\mathbf{d} - \mathbf{a} = \mathbf{c} - \mathbf{b}, \qquad \text{that is, } \overrightarrow{AD} = \overrightarrow{BC}$$

   and the quadrilateral $ABCD$ is a parallelogram if and only if $\overrightarrow{AD} = \overrightarrow{BC}$ (the sides AD and BC are parallel, equal in length, and point in the same direction). (So the statement is true for every choice of P.)

Recommended for students from vocational secondary schools

6. In rectangle $ABCD$, $AB = 2a$ and $BC = a$. Circles are drawn on the sides AB and AD as diameters; their intersection point inside the rectangle is M. Show that M is on the diagonal BD. How long are the segments MA, MB, and MD? (13 points)

   Solution: By Thales' theorem, $\angle AMB = 90^\circ$ (since M is on the circle with diameter AB) and $\angle AMD = 90^\circ$ (since M is on the circle with diameter AD). So both MB and MD are perpendicular to MA at M, so B, M, D are on one line: M is on the diagonal BD, and it is the foot of the perpendicular from A to BD.



$BD = \sqrt{4a^2 + a^2} = \sqrt{5}a$. In the right triangle ABD, the altitude is AM, so (computing the area in two ways, and using the similar triangles)

$$MA = \frac{AB \cdot AD}{BD} = \frac{2a^2}{\sqrt{5}a} = \frac{2\sqrt{5}}{5}a, \qquad MB = \frac{AB^2}{BD} = \frac{4\sqrt{5}}{5}a, \qquad MD = \frac{AD^2}{BD} = \frac{\sqrt{5}}{5}a$$

7. Solve the given equation over the real numbers.

$$x^2 + 2x \sin xy + 1 = 0$$

(14 points)

Solution: Since $1 = \sin^2 xy + \cos^2 xy$, the equation can be written as

$$(x + \sin xy)^2 + \cos^2 xy = 0$$

A sum of two squares is zero only if both are zero: $\cos xy = 0$ and $x = -\sin xy$. Then $\sin xy = \pm 1$, so $x = \pm 1$.

If $x = 1$: $\sin y = -1$. If $x = -1$: $\sin(-y) = 1$, that is, $\sin y = -1$ again. The solutions are

$$x = 1 \text{ or } x = -1, \qquad y = -\frac{\pi}{2} + 2k\pi, \qquad k \text{ is an integer}$$

8. Prove that if the sum of the real numbers a and b is 1, then

$$a^4 + b^4 \geq \frac{1}{8}$$

(15 points)

Solution: For any real numbers u and v, $u^2 + v^2 \geq \frac{(u+v)^2}{2}$, because this is equivalent to $(u-v)^2 \geq 0$. Using this twice,

$$a^4 + b^4 \geq \frac{\left(a^2 + b^2\right)^2}{2} \geq \frac{1}{2}\left(\frac{(a+b)^2}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Equality holds for $a = b = \frac{1}{2}$.

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