translated from: KoMaL, September 1988, pages 251-252
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These problems were given at the joint written final (érettségi) and entrance examination in mathematics in Hungary in 1988. Problems 1–5 were the same for all students; for problems 6–8, there was one version (A) recommended for students from academic high schools and another one (B) for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all students

1. Solve the given system of equations over the real numbers.

$$\frac{4}{x}+\frac{5}{y}=23, \qquad \frac{5}{x}-\frac{4}{y}=-2$$

(9 points)

Solution: Let $u=\dfrac{1}{x}$ and $v=\dfrac{1}{y}$. Then $4u+5v=23$ and $5u-4v=-2$. Multiplying the first equation by 4 and the second one by 5 and adding them, we get $41u=82$, so $u=2$ and then $5v=23-8=15$, $v=3$. So

$$x=\frac{1}{2}, \qquad y=\frac{1}{3}$$

2. Solve the given equation over the real numbers.

$$\left(\frac{3}{4}\right)^{\log x}+\left(\frac{4}{3}\right)^{\log x}=\frac{25}{12}$$

(11 points)

Solution: We need $x>0$. Let $t=\left(\dfrac{3}{4}\right)^{\log x}>0$. Then $\left(\dfrac{4}{3}\right)^{\log x}=\dfrac{1}{t}$, so

$$t+\frac{1}{t}=\frac{25}{12}, \qquad 12t^2-25t+12=0, \qquad (4t-3)\,(3t-4)=0$$

If $t=\dfrac{3}{4}$, then $\log x=1$ and $x=10$. If $t=\dfrac{4}{3}$, then $\log x=-1$ and $x=\dfrac{1}{10}$. The solutions are $x=10$ and $x=\dfrac{1}{10}$.

3. A rhombus with a 30° angle is drawn around a circle of area t. Express the area of the rhombus in terms of t. (12 points)

Solution: Let r be the radius of the circle, so $t=r^2\pi$. The height of the rhombus is the diameter of the circle, $2r$. If the side of the rhombus is a, then its height is $a\sin 30^\circ=\dfrac{a}{2}$, so $a=4r$. The area of the rhombus is

$$a\cdot 2r=8r^2=\frac{8t}{\pi}$$

4. For which real numbers x are the following expressions defined?

a) $\sqrt{3-\sqrt{x-2}}$ b) $\sqrt{x-\sqrt{x-2}}$ c) $\sqrt{1-\tan^2 x}$

(13 points)

Solution: a) We need $x \geq 2$ and $\sqrt{x-2} \leq 3$, that is, $x-2 \leq 9$. The expression is defined for $2 \leq x \leq 11$.

b) We need $x \geq 2$ and $\sqrt{x-2} \leq x$. For $x \geq 2$ both sides are nonnegative, so this is equivalent to $x-2 \leq x^2$, that is, $x^2 - x + 2 \geq 0$. This is true for every x, because $x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} > 0$. The expression is defined for $x \geq 2$.

c) We need $\tan x$ to be defined and $\tan^2 x \leq 1$, that is, $-1 \leq \tan x \leq 1$. The expression is defined for

$$-\frac{\pi}{4} + k\pi \leq x \leq \frac{\pi}{4} + k\pi, \qquad k \text{ is an integer}$$

5. In triangle ABC, the angle at vertex A is twice the angle at vertex B, twice the side BC is equal to three times the side AC, and the area of the triangle is 27 times the sine of the angle at vertex C. Find the angles and the sides of the triangle. (13 points)

Solution: Let $a = BC$, $b = AC$, $c = AB$, and let β be the angle at B, so the angle at A is 2β. By the Law of Sines,

$$\frac{a}{b} = \frac{\sin 2\beta}{\sin \beta} = 2\cos\beta$$

and $\frac{a}{b} = \frac{3}{2}$, so $\cos\beta = \frac{3}{4}$. The area is $\frac{1}{2}ab\sin\gamma = 27\sin\gamma$, so $ab = 54$. With $a = \frac{3}{2}b$ this gives $b^2 = 36$, $b = 6$, and $a = 9$. The angle at C is $\gamma = 180^\circ - 3\beta$, so again by the Law of Sines,

$$\frac{c}{b} = \frac{\sin 3\beta}{\sin\beta} = 3 - 4\sin^2\beta = 3 - 4\cdot\frac{7}{16} = \frac{5}{4}$$

(using $\sin 3\beta = 3\sin\beta - 4\sin^3\beta$), so $c = 7.5$. The sides are

$$BC = 9, \qquad AC = 6, \qquad AB = 7.5$$

and the angles are $\beta = \cos^{-1}\frac{3}{4} \approx 41.41^\circ$ at B, $2\beta \approx 82.82^\circ$ at A, and $180^\circ - 3\beta \approx 55.77^\circ$ at C.

A. Recommended for students from academic high schools

6. Find the equation of the circle whose center is on the x-axis, and which is tangent from the inside to the circle k_1: $x^2 + y^2 + 4x + 6y - 87 = 0$, and tangent from the outside to the circle k_2: $x^2 + y^2 - 20x - 12y + 111 = 0$. (13 points)

Solution: Completing the squares,

$$k_1:\ (x+2)^2 + (y+3)^2 = 100, \qquad k_2:\ (x-10)^2 + (y-6)^2 = 25$$

so k_1 has center $O_1\,(-2,-3)$ and radius 10, and k_2 has center $O_2\,(10,6)$ and radius 5. Let the center of the circle be K and its radius r. The circle is inside k_1 and touches it, so $O_1K = 10 - r$, and it touches k_2 from the outside, so $O_2K = 5 + r$. Then

$$O_1K + KO_2 = 15 = \sqrt{12^2 + 9^2} = O_1O_2$$

so K is on the segment O_1O_2. (So k_1 and k_2 touch each other, and the circle we are looking for fits between them.) The points of the segment are $(-2 + 12s, -3 + 9s)$ with $0 \leq s \leq 1$, and K is on the x-axis, so $s = \frac{1}{3}$ and $K = (2, 0)$. Then $r = 10 - O_1K = 10 - 5 = 5$, and $O_2K = 10 = 5 + r$ checks. The circle is

$$(x-2)^2 + y^2 = 25$$

7. The base of a pyramid is a square, and its lateral faces are congruent isosceles triangles. The height of the pyramid makes a 30° angle with the lateral faces. A cube is placed inside the pyramid so that four vertices of the cube are on the base of the pyramid, and the other four vertices are on the lateral faces, on the altitudes of the lateral faces drawn from the apex of the pyramid. Find the ratio of the volumes of the pyramid and the cube. (14 points)

Solution: Let the side of the base be 2 and let the base be centered at the origin with sides parallel to the axes. The altitudes of the lateral faces from the apex go from the apex to the midpoints $(\pm 1, 0)$, $(0, \pm 1)$ of the base edges. The height m of the pyramid makes 30° with the lateral faces, so the lateral faces make 60° with the base, and $m = 1 \cdot \tan 60^\circ = \sqrt{3}$.

The four upper vertices of the cube are on these four altitudes, so the cube is turned 45° compared to the base: seen from above, its vertices are on the lines $x = 0$ and $y = 0$ (which join the midpoints of opposite base edges), at distance $\dfrac{x}{\sqrt{2}}$ from the center, where x is the edge of the cube. The upper vertices are at height x, where the cross-section of the pyramid is a square of half-side $1 - \dfrac{x}{\sqrt{3}}$ (the pyramid shrinks linearly to the apex), and the upper vertices are the midpoints of the sides of this square. So

$$\frac{x}{\sqrt{2}} = 1 - \frac{x}{\sqrt{3}}, \qquad x = \frac{1}{\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}} = \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} = \sqrt{6}\left(\sqrt{3}-\sqrt{2}\right) = 3\sqrt{2} - 2\sqrt{3} \approx 0.779$$

The volume of the pyramid is $\dfrac{1}{3} \cdot 4 \cdot \sqrt{3} = \dfrac{4\sqrt{3}}{3}$, and $\dfrac{1}{x^3} = \dfrac{\left(\sqrt{3}+\sqrt{2}\right)^3}{6\sqrt{6}} = \dfrac{9\sqrt{3}+11\sqrt{2}}{6\sqrt{6}}$. So the ratio of the volumes is

$$\frac{V_{\text{pyramid}}}{V_{\text{cube}}} = \frac{4\sqrt{3}}{3} \cdot \frac{9\sqrt{3}+11\sqrt{2}}{6\sqrt{6}} = \frac{2\left(9\sqrt{3}+11\sqrt{2}\right)}{9\sqrt{2}} = \sqrt{6} + \frac{22}{9} \approx 4.894$$

8. Let p and q be positive integers with $q \geq p$. Prove that

$$\frac{q+\dfrac{1}{2}}{p-\dfrac{1}{2}} > \frac{1}{p} + \frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{q}$$

(15 points)

Solution: The right-hand side has $q - p + 1$ terms, and each of them is at most $\dfrac{1}{p}$, so the right-hand side is at most $\dfrac{q-p+1}{p}$. The left-hand side is greater than $\dfrac{q+\dfrac{1}{2}}{p}$, because $0 < p - \dfrac{1}{2} < p$. Finally, $q + \dfrac{1}{2} > q - p + 1$, because $p \geq 1 > \dfrac{1}{2}$. So

$$\frac{q+\dfrac{1}{2}}{p-\dfrac{1}{2}} > \frac{q+\dfrac{1}{2}}{p} > \frac{q-p+1}{p} \geq \frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{q}$$

Remark: The inequality is true in a much stronger form too: the sum on the right is less than $\ln \dfrac{q+\dfrac{1}{2}}{p-\dfrac{1}{2}}$. This follows from the fact that each term $\dfrac{1}{k}$ is less than the area under the graph of $y = \dfrac{1}{x}$ between $k - \dfrac{1}{2}$ and $k + \dfrac{1}{2}$, because the graph is convex.

B. Recommended for students from vocational secondary schools

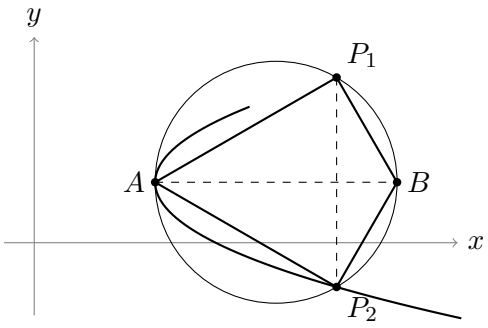
6. From the points P_1 and P_2 of the parabola $x-2=(y-1)^2$, the segment with endpoints $A\,(2,1)$ and $B\,(6,1)$ is seen at a right angle. Find the area of the quadrilateral AP_1BP_2. (13 points)

   Solution: The points from which AB is seen at a right angle are on the circle with diameter AB (Thales' theorem), whose center is $(4,1)$ and radius is 2: $(x-4)^2+(y-1)^2=4$. Let $u=y-1$; on the parabola $x=2+u^2$, so

   $$\left(u^2-2\right)^2+u^2=4, \qquad u^4-3u^2=0$$

   $u=0$ gives the point A itself (the vertex of the parabola), and $u=\pm\sqrt{3}$ gives $P_1\left(5,1+\sqrt{3}\right)$ and $P_2\left(5,1-\sqrt{3}\right)$. The diagonals of the quadrilateral AP_1BP_2 are $AB=4$ and $P_1P_2=2\sqrt{3}$, and they are perpendicular, so its area is

   $$\frac{4\cdot 2\sqrt{3}}{2}=4\sqrt{3}\approx 6.93$$



7. A right circular cylinder has height 2 and radius 7. A square is inscribed in the cylinder so that its plane intersects the axis of the cylinder, and its vertices are on the circles of the base and of the top. Find the side of the square and the angle between the axis of the cylinder and the plane of the square. (14 points)

   Solution: Two vertices of the square are on the base circle and two on the top circle, so two opposite sides of the square are horizontal chords of these circles. By symmetry, these chords are at the same distance d from the axis, on opposite sides of it. If the side of the square is s, then a chord gives $s^2=4\left(49-d^2\right)$. A side going from the base to the top rises 2 units, while its projection to the base is perpendicular to the chords and has length $2d$, so $s^2=4d^2+4$. Then

   $$4\left(49-d^2\right)=4d^2+4, \qquad d^2=24, \qquad s^2=100, \qquad s=10$$

   The angle φ between the axis and the plane of the square is the angle between the axis and a side going from the base to the top (this side is the steepest line in the plane). So

   $$\cos\varphi=\frac{2}{s}=\frac{1}{5}, \qquad \varphi\approx 78.46^\circ$$

8. Solve the given inequality over the real numbers.

$$2\sqrt{|1-2x|} \geq 1+x$$

(15 points)

Solution: If $x < -1$, then the right-hand side is negative and the inequality is true. If $x \geq -1$, both sides are nonnegative, and we can square: $4\,|1-2x| \geq (1+x)^2$.

If $-1 \leq x \leq \frac{1}{2}$: $4-8x \geq x^2+2x+1$, that is, $x^2+10x-3 \leq 0$, so $-5-2\sqrt{7} \leq x \leq -5+2\sqrt{7}$. Together with the condition: $-1 \leq x \leq 2\sqrt{7}-5$ (note that $2\sqrt{7}-5 \approx 0.29 < \frac{1}{2}$).

If $x > \frac{1}{2}$: $8x-4 \geq x^2+2x+1$, that is, $x^2-6x+5 \leq 0$, so $1 \leq x \leq 5$.

The solution is

$$x \leq 2\sqrt{7}-5 \approx 0.29 \quad \text{or} \quad 1 \leq x \leq 5$$

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