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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following over the real numbers.

$$\text{a) } \frac{4x}{|x-1|} = 2x \qquad \text{b) } \frac{x+2}{2x-2} > -3$$

Solution: a) We need $x \neq 1$. The equation is $2x \left(\frac{2}{|x-1|} - 1 \right) = 0$, so $x = 0$, or $|x-1| = 2$, that is, $x = 3$ or $x = -1$. The solutions are $x = -1$, $x = 0$, and $x = 3$.

b) We need $x \neq 1$. Adding 3 to both sides:

$$\frac{x+2+3(2x-2)}{2x-2} > 0, \qquad \frac{7x-4}{2(x-1)} > 0$$

The fraction is positive if the numerator and the denominator have the same sign. The solution is $x < \frac{4}{7}$ or $x > 1$.

2. The sides of a triangle form a geometric sequence with ratio $\frac{4}{3}$. Find the largest angle of the triangle.

Solution: The sides are $a, \frac{4}{3}a, \frac{16}{9}a$; the shape of the triangle does not depend on a , so we can choose $a = 9$: the sides are 9, 12, 16. (This is a triangle, since $9 + 12 > 16$.) The largest angle γ is opposite the longest side. By the Law of Cosines,

$$\cos \gamma = \frac{9^2 + 12^2 - 16^2}{2 \cdot 9 \cdot 12} = -\frac{31}{216}, \qquad \gamma \approx 98.25^\circ$$

3. Find the equation of the circle that passes through the centers of the following three circles.

$$x^2 + y^2 = 4; \quad 3x^2 - 18x + 3y^2 + 12y = 36; \quad x^2 + y^2 + 10x + 4y + 12 = 0$$

Solution: The first circle is centered at $(0, 0)$. Dividing the second equation by 3 and completing the squares: $(x - 3)^2 + (y + 2)^2 = 25$, centered at $(3, -2)$. The third one is $(x + 5)^2 + (y + 2)^2 = 17$, centered at $(-5, -2)$.

The center of the circle we are looking for is on the perpendicular bisector of the segment joining $(3, -2)$ and $(-5, -2)$, which is the line $x = -1$. So the center is $(-1, k)$, and it is equally far from $(0, 0)$ and $(3, -2)$:

$$1 + k^2 = 16 + (k + 2)^2, \quad 4k = -19, \quad k = -\frac{19}{4}$$

The radius squared is $1 + \frac{361}{16} = \frac{377}{16}$, so the circle is

$$(x + 1)^2 + \left(y + \frac{19}{4}\right)^2 = \frac{377}{16}, \quad \text{or} \quad 2x^2 + 2y^2 + 4x + 19y = 0$$

4. For which real numbers x and y are both of the following equations true?

$$\sqrt{5^{x-y}} = \left(\frac{1}{5}\right)^{x-2y}; \quad \log_4(x+y) + \log_4(x-y) = 2$$

Solution: We need $x + y > 0$ and $x - y > 0$. The first equation says $5^{\frac{x-y}{2}} = 5^{-(x-2y)}$, so

$$\frac{x-y}{2} = -x + 2y, \quad 3x = 5y$$

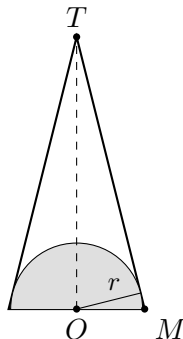
The second equation says $(x+y)(x-y) = 4^2$, that is, $x^2 - y^2 = 16$. With $y = \frac{3}{5}x$: $\frac{16}{25}x^2 = 16$, so $x = \pm 5$. If $x = -5$, then $y = -3$ and $x + y < 0$, which is not allowed. The only solution is

$$x = 5, \quad y = 3$$

5. The base edge of a right pyramid with a square base is 4 units, and its height is 8 units. We place a hemisphere into the pyramid so that the circle bounding the hemisphere is in the base of the pyramid, and the hemisphere touches the lateral faces of the pyramid. Find the ratio of the volumes of the pyramid and the hemisphere.

Solution: By symmetry, the center of the hemisphere is the center O of the base. Consider the cross-section through the apex T perpendicular to a pair of base edges: it is an isosceles triangle with base 4 and height 8, and the hemisphere appears as a half disk centered at O , tangent to the sides of this triangle. Let M be the midpoint of a base edge; the radius r is the distance of O from TM , which is the altitude to the hypotenuse of the right triangle OMT with legs $OM = 2$, $OT = 8$, and hypotenuse $TM = \sqrt{68} = 2\sqrt{17}$:

$$r = \frac{OM \cdot OT}{TM} = \frac{16}{2\sqrt{17}} = \frac{8}{\sqrt{17}} \approx 1.94$$



The volume of the pyramid is $\frac{1}{3} \cdot 16 \cdot 8 = \frac{128}{3}$, and the volume of the hemisphere is $\frac{2}{3}r^3\pi = \frac{2}{3} \cdot \frac{512}{17\sqrt{17}}\pi$. So the ratio is

$$\frac{V_{\text{pyramid}}}{V_{\text{hemisphere}}} = \frac{128}{3} \cdot \frac{3 \cdot 17\sqrt{17}}{1024\pi} = \frac{17\sqrt{17}}{8\pi} \approx 2.79$$

Option A

6/A. Consider the parabola $y = 4 - x^2$, and all the rectangles that have one side on the x -axis and the two opposite vertices on the parabola. Find the sides of the rectangle with the largest perimeter.

Solution: By symmetry, the vertices on the parabola are $(-a, 4 - a^2)$ and $(a, 4 - a^2)$ with $0 < a < 2$, so the sides are $2a$ and $4 - a^2$. The perimeter is

$$2 \cdot 2a + 2(4 - a^2) = -2a^2 + 4a + 8 = 10 - 2(a - 1)^2$$

which is largest for $a = 1$. The rectangle with the largest perimeter has sides 2 (on the x -axis) and 3, and its perimeter is 10.

7/A. We draw the median AP and the interior angle bisector AQ of triangle ABC , where P and Q are on the side BC . The circle through A, P, Q intersects the side AB at E and the side AC at F . Prove that $BE = CF$.

Solution: Let $a = BC, b = CA, c = AB$. The point P is the midpoint, so $BP = CP = \frac{a}{2}$, and by the angle bisector theorem $BQ = \frac{ac}{b+c}$ and $CQ = \frac{ab}{b+c}$.

The point B is outside the circle, and two lines through B intersect the circle at E, A and at Q, P . By the power of a point theorem (secant–secant theorem),

$$BE \cdot BA = BQ \cdot BP, \quad BE = \frac{1}{c} \cdot \frac{ac}{b+c} \cdot \frac{a}{2} = \frac{a^2}{2(b+c)}$$

Similarly from C :

$$CF \cdot CA = CQ \cdot CP, \quad CF = \frac{1}{b} \cdot \frac{ab}{b+c} \cdot \frac{a}{2} = \frac{a^2}{2(b+c)}$$

So $BE = CF$. (If $b = c$, then $P = Q$ and the circle is tangent to BC at P ; the same computation works with $BE \cdot BA = BP^2$ and $CF \cdot CA = CP^2$.)

8/A. Solve the given equation over the real numbers.

$$\cot \frac{2\pi x}{x^2 + x + 1} = \frac{-\sqrt{3}}{3}$$

Solution: $x^2 + x + 1 > 0$ for every x . Let $t = \frac{x}{x^2 + x + 1}$. First we find the possible values of t : the equation $tx^2 + (t-1)x + t = 0$ has a real solution x exactly when $t = 0$ or its discriminant is not negative:

$$(t-1)^2 - 4t^2 \geq 0, \quad (1-3t)(1+t) \geq 0, \quad -1 \leq t \leq \frac{1}{3}$$

$\cot \alpha = -\frac{\sqrt{3}}{3}$ means $\alpha = \frac{2\pi}{3} + k\pi$, so $2\pi t = \frac{2\pi}{3} + k\pi$, that is, $t = \frac{1}{3} + \frac{k}{2}$. In the interval $\left[-1, \frac{1}{3}\right]$ this gives $t = \frac{1}{3}$,
 $t = -\frac{1}{6}, t = -\frac{2}{3}$.

$$\begin{aligned} t = \frac{1}{3} : \quad x^2 + x + 1 &= 3x, \quad (x-1)^2 = 0 & x &= 1 \\ t = -\frac{1}{6} : \quad x^2 + x + 1 &= -6x, \quad x^2 + 7x + 1 = 0 & x &= \frac{-7 \pm 3\sqrt{5}}{2} \\ t = -\frac{2}{3} : \quad 2x^2 + 2x + 2 &= -3x, \quad 2x^2 + 5x + 2 = 0 & x &= -\frac{1}{2} \text{ or } x = -2 \end{aligned}$$

The solutions are $x = 1, x = -\frac{1}{2}, x = -2$, and $x = \frac{-7 \pm 3\sqrt{5}}{2}$.

Option B

6/B. For what value of p is the sum of the squares of the roots of the equation $x^2 + (3p-2)x - 7p-1 = 0$ the smallest?

Solution: The discriminant is $(3p-2)^2 + 4(7p+1) = 9p^2 + 16p + 8$, which is always positive (its own discriminant is $256 - 288 < 0$), so the equation always has two real roots. By Viète's formulas, $x_1 + x_2 = 2 - 3p$ and $x_1 x_2 = -7p - 1$, so

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = 9p^2 + 2p + 6 = 9\left(p + \frac{1}{9}\right)^2 + \frac{53}{9}$$

This is smallest for $p = -\frac{1}{9}$, and the smallest value is $\frac{53}{9}$.

7/B. A circle with center C is intersected by a parabola whose vertex is the center of the circle and whose focus F is on the circle. Let A and B be the intersection points of the parabola and the circle, and let E be the intersection point of the line AB and the axis of symmetry of the parabola. Show that the segment DE is the geometric mean of the segment FE and the diameter, where D is the point of the circle opposite to F .

Solution: Let the radius be r , and use coordinates with C at the origin and $F(0, r)$. The parabola with vertex $(0, 0)$ and focus $(0, r)$ is $x^2 = 4ry$, and the circle is $x^2 + y^2 = r^2$. At the intersection points,

$$4ry + y^2 = r^2, \quad y = (\sqrt{5} - 2)r$$

(the other root is negative, and $y \geq 0$ on the parabola). So AB is the horizontal line $y = (\sqrt{5} - 2)r$, and $E = (0, (\sqrt{5} - 2)r)$. With $D = (0, -r)$:

$$FE = r - (\sqrt{5} - 2)r = (3 - \sqrt{5})r, \quad DE = (\sqrt{5} - 2)r + r = (\sqrt{5} - 1)r$$

and

$$DE^2 = (6 - 2\sqrt{5})r^2 = (3 - \sqrt{5})r \cdot 2r = FE \cdot 2r$$

so DE is the geometric mean of FE and the diameter $2r$.

8/B. For which positive integers N is $N + 1987$ a divisor of $N^2 + 1987$?

Solution: Since

$$N^2 + 1987 = (N + 1987)(N - 1987) + 1987^2 + 1987 = (N + 1987)(N - 1987) + 1987 \cdot 1988$$

$N + 1987$ divides $N^2 + 1987$ if and only if it divides $1987 \cdot 1988$. Here 1987 is a prime number and $1988 = 2^2 \cdot 7 \cdot 71$. So we need the divisors d of $1987 \cdot 1988$ with $d > 1987$ (because $N = d - 1987 \geq 1$).

A divisor that is not a multiple of 1987 is a divisor of 1988; the only one greater than 1987 is 1988, giving $N = 1$. A multiple of 1987 is $1987m$ with m a divisor of 1988, and $m > 1$, giving $N = 1987(m - 1)$. The divisors of 1988 are 1, 2, 4, 7, 14, 28, 71, 142, 284, 497, 994, 1988, so there are 12 solutions:

$$N = 1 \quad \text{and} \quad N = 1987k \quad \text{for} \quad k = 1, 3, 6, 13, 27, 70, 141, 283, 496, 993, 1987$$