translated from: KoMaL, September 1988, pages 252-253
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These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1988. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. Find the exact values of the given expressions without using approximate values.

$$A = 10^{1-\log 2} + 3^{\log_9 36}, \qquad B = 125^{-\frac{1}{3}} \cdot \sqrt[3]{1000} \cdot \log_{\sin\frac{\pi}{6}} \sqrt{2}, \qquad C = \sqrt{\log\tan 36^\circ + \log\tan 54^\circ}$$

(9 points)

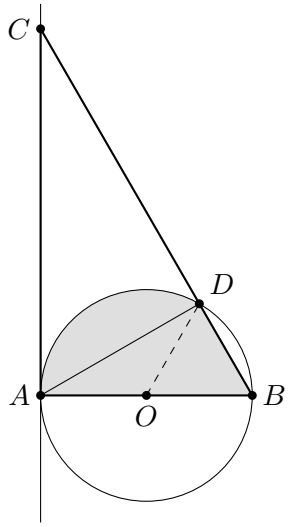
Solution: $10^{1-\log 2} = \dfrac{10}{10^{\log 2}} = \dfrac{10}{2} = 5$, and $3^{\log_9 36} = 3^{\log_3 6} = 6$ (because $\log_9 36 = \dfrac{\log_3 36}{\log_3 9} = \dfrac{2\log_3 6}{2}$). So $A = 11$.

$125^{-\frac{1}{3}} = \dfrac{1}{5}$, $\sqrt[3]{1000} = 10$, and $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$, $\log_{\frac{1}{2}}\sqrt{2} = -\dfrac{1}{2}$ (because $\left(\dfrac{1}{2}\right)^{-\frac{1}{2}} = \sqrt{2}$). So $B = \dfrac{1}{5}\cdot 10\cdot\left(-\dfrac{1}{2}\right) = -1$.

$\log\tan 36^\circ + \log\tan 54^\circ = \log\left(\tan 36^\circ \tan 54^\circ\right) = \log 1 = 0$, because $\tan 54^\circ = \tan\left(90^\circ - 36^\circ\right) = \dfrac{1}{\tan 36^\circ}$. So $C = 0$.

2. Let AB be a diameter of a circle of radius 1, and let C be the point on the tangent line to the circle at A for which the segment AC has length $2\sqrt{3}$. Find the area of the common part of the triangle ABC and the disk. (11 points)

Solution: The triangle ABC has a right angle at A, with $AB = 2$ and $AC = 2\sqrt{3}$, so $\tan\angle ABC = \sqrt{3}$ and $\angle ABC = 60^\circ$. Let D be the second intersection point of BC with the circle. Since AB is a diameter, $\angle ADB = 90^\circ$, so $BD = AB\cos 60^\circ = 1$ and $AD = AB\sin 60^\circ = \sqrt{3}$.



The common part consists of the triangle ABD and the circular segment cut off by the chord AD. The area of the triangle ABD is $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$. The central angle over the chord AD is $\angle AOD = 2\angle ABD = 120°$, so the area of the segment is

$$\frac{1}{2}\left(\frac{2\pi}{3} - \sin 120°\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

The area of the common part is

$$\frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{4} = \frac{\pi}{3} + \frac{\sqrt{3}}{4} \approx 1.48$$

3. We build a large cube out of small cubes with unit edges. If we build the largest possible cube from the small cubes we have, then 107 small cubes are left over. If we want one more small cube along each edge, then we would need 62 more small cubes. How many small cubes do we have? (12 points)

Solution: If the largest cube has n small cubes along each edge, then the number of small cubes is

$$n^3 + 107 = (n+1)^3 - 62, \qquad 3n^2 + 3n + 1 = 169, \qquad n^2 + n - 56 = 0$$

so $(n+8)(n-7) = 0$ and $n = 7$. We have $7^3 + 107 = 450$ small cubes. (Check: $8^3 - 62 = 512 - 62 = 450$.)

4. For which pairs of real numbers is the following system of equations true?

$$\begin{aligned} y - x &= 44 \\ \sqrt{\frac{6x}{x+y}} + \sqrt{\frac{x+y}{6x}} &= \frac{5}{2} \end{aligned}$$

(13 points)

Solution: Let $t = \sqrt{\frac{6x}{x+y}} > 0$. Then

$$t + \frac{1}{t} = \frac{5}{2}, \qquad 2t^2 - 5t + 2 = 0, \qquad t = 2 \text{ or } t = \frac{1}{2}$$

If $t = 2$: $\frac{6x}{x+y} = 4$, so $6x = 4x + 4y$, $x = 2y$. Then $y - 2y = 44$, so $y = -44$ and $x = -88$.

If $t = \frac{1}{2}$: $\frac{6x}{x+y} = \frac{1}{4}$, so $24x = x + y$, $y = 23x$. Then $22x = 44$, so $x = 2$ and $y = 46$.

Both pairs work (in the first one, $\frac{6x}{x+y} = \frac{-528}{-132} = 4$). The solutions are $(x, y) = (-88, -44)$ and $(2, 46)$.

5. The line $x+y=30$ intersects the x-axis at A and the y-axis at B, and let O be the origin. Find the equation of the line e that is parallel to the line $x-3y=6$, intersects the segment OB at C and the segment AB at D, and for which the area of the quadrilateral $OADC$ is 234 units. (13 points)

Solution: $A=(30,0)$ and $B=(0,30)$. The line e is parallel to $x-3y=6$, so its equation is $x-3y=-3k$ for some number k; then $C=(0,k)$, with $0<k<30$. Solving $x+y=30$ and $x-3y=-3k$: $D=\left(\dfrac{90-3k}{4},\dfrac{30+3k}{4}\right)$.
The quadrilateral $OADC$ is made of the triangles OAD and ODC:

$$\frac{1}{2}\cdot 30\cdot\frac{30+3k}{4}+\frac{1}{2}\cdot k\cdot\frac{90-3k}{4}=234$$

Multiplying by 8:

$$900+90k+90k-3k^2=1872,\qquad k^2-60k+324=0,\qquad (k-6)\,(k-54)=0$$

Since $k<30$, $k=6$. Then $C=(0,6)$, $D=(18,12)$, and the equation of e is

$$x-3y=-18$$

Recommended for students from academic high schools

6. For given real numbers p and q, find the first term and the difference of every arithmetic sequence in which the sum of the first four terms is p, and the quotient of the fourth term and the first term is q. (13 points)

Solution: Let the first term be $a\neq 0$ and the difference d. Then

$$4a+6d=p,\qquad \frac{a+3d}{a}=q$$

The second equation gives $3d=(q-1)\,a$, and then the first one becomes

$$4a+2\,(q-1)\,a=p,\qquad 2\,(q+1)\,a=p$$

If $q\neq -1$ and $p\neq 0$: there is exactly one such sequence,

$$a=\frac{p}{2\,(q+1)},\qquad d=\frac{(q-1)\,p}{6\,(q+1)}$$

If $q\neq -1$ and $p=0$: then $a=0$, which is not allowed (the quotient is not defined), so there is no such sequence.

If $q=-1$ and $p\neq 0$: the equation $0\cdot a=p$ has no solution, so there is no such sequence.

If $q=-1$ and $p=0$: every $a\neq 0$ works, with $d=-\dfrac{2}{3}a$. (For example 3, 1, -1, -3.)

7. Solve the given equation, if p is a prime number and n is a positive integer.

$$1+p+p^2+\cdots+p^n=2801$$

(14 points)

Solution: Subtracting 1:

$$p\left(1+p+\cdots+p^{n-1}\right)=2800=2^4\cdot 5^2\cdot 7$$

so p is 2, 5, or 7. If $p=2$, then the left-hand side of the original equation is $2^{n+1}-1$, which is never 2801, because 2802 is not a power of 2. If $p=5$: $1+5+25+125+625=781$ and $781+3125=3906$, so the sum skips 2801. If $p=7$: $1+7+49+343+2401=2801$. The only solution is $p=7$, $n=4$.

8. Consider four vertices of a unit cube such that no two of them are on the same edge of the cube. The tetrahedron determined by these four points is cut by planes parallel to two opposite edges of the tetrahedron. Considering all such planes, what are the largest possible perimeter and the largest possible area of the cross-section? (15 points)

Solution: The four vertices are pairwise endpoints of face diagonals of the cube, so they form a regular tetrahedron with edge $\sqrt{2}$. Let the plane be parallel to the opposite edges AB and CD. It cuts the edges AC, AD, BD, BC at K, L, M, N. If $\dfrac{AK}{AC} = \lambda$ with $0 < \lambda < 1$, then (by the intercept theorem) $KL \parallel CD$ with $KL = \lambda\sqrt{2}$, and $KN \parallel AB$ with $KN = (1-\lambda)\sqrt{2}$, and similarly for the other two sides. So the cross-section is a parallelogram with sides $\lambda\sqrt{2}$ and $(1-\lambda)\sqrt{2}$. Opposite edges of a regular tetrahedron are perpendicular (in the cube, AB and CD are diagonals of opposite faces in perpendicular directions), so the cross-section is a rectangle.

Its perimeter is $2\left(\lambda\sqrt{2} + (1-\lambda)\sqrt{2}\right) = 2\sqrt{2}$, the same for every such plane, so its largest value is $2\sqrt{2}$.

Its area is $2\lambda(1-\lambda)$, and by the inequality between the arithmetic and geometric means,

$$\lambda(1-\lambda) \le \left(\frac{\lambda + (1-\lambda)}{2}\right)^2 = \frac{1}{4}$$

with equality for $\lambda = \dfrac{1}{2}$. So the largest area is $\dfrac{1}{2}$, for the plane through the midpoints of the edges: the cross-section is then a square with side $\dfrac{\sqrt{2}}{2}$.

Recommended for students from vocational secondary schools

6. The difference of the volumes of two right circular cylinders is 48π, and the difference of their lateral surface areas is 24π. The radius of the cylinder with the larger volume is 1 more than the radius of the other one, and its height is $\dfrac{4}{3}$ times the height of the other one. Find the heights and the radii of the cylinders. (13 points)

Solution: Let the smaller cylinder have radius r and height h; then the larger one has radius $r+1$ and height $\dfrac{4}{3}h$. So

$$\pi(r+1)^2 \cdot \frac{4}{3}h - \pi r^2 h = 48\pi, \qquad 2\pi(r+1) \cdot \frac{4}{3}h - 2\pi r h = 24\pi$$

Multiplying by $\dfrac{3}{\pi}$ and by $\dfrac{3}{2\pi}$:

$$h\left(r^2 + 8r + 4\right) = 144, \qquad h(r+4) = 36$$

Dividing the first equation by the second one:

$$\frac{r^2 + 8r + 4}{r+4} = 4, \qquad r^2 + 4r - 12 = 0, \qquad (r+6)(r-2) = 0$$

so $r = 2$ and $h = \dfrac{36}{6} = 6$. The smaller cylinder has radius 2 and height 6, the larger one has radius 3 and height 8. (Check: $72\pi - 24\pi = 48\pi$ and $48\pi - 24\pi = 24\pi$.)

7. Solve the given equation over the pairs of real numbers.

$$2\cos^2\frac{x^2+3y}{6}=3^x+3^{-x}$$

(14 points)

Solution: The left-hand side is at most 2. By the inequality between the arithmetic and geometric means, the right-hand side is at least $2\sqrt{3^x\cdot 3^{-x}}=2$, with equality only if $3^x=3^{-x}$, that is, $x=0$. So both sides must be 2: $x=0$ and $\cos^2\frac{y}{2}=1$, that is, $\frac{y}{2}=k\pi$. The solutions are

$$x=0,\qquad y=2k\pi,\qquad k \text{ is an integer}$$

8. In a geometric sequence of positive numbers, the sum of the first n terms is S, and the sum of the reciprocals of the first n terms is R. Express the product of the first n terms in terms of S, R, and n. (15 points)

Solution: Let the terms be $a, aq, \ldots, aq^{n-1}$ with $a>0$, $q>0$. Then

$$R=\frac{1}{a}+\frac{1}{aq}+\cdots+\frac{1}{aq^{n-1}}=\frac{q^{n-1}+q^{n-2}+\cdots+1}{aq^{n-1}}=\frac{S}{a^2q^{n-1}}$$

because $q^{n-1}+\cdots+1=\frac{S}{a}$. So $a^2q^{n-1}=\frac{S}{R}$. The product of the terms is

$$a^nq^{1+2+\cdots+(n-1)}=a^nq^{\frac{n(n-1)}{2}}=\left(a^2q^{n-1}\right)^{\frac{n}{2}}=\left(\frac{S}{R}\right)^{\frac{n}{2}}$$

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