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translated from: KoMaL, September 1989, pages 253-254

Pdf file: KoMaL, September 1989, pages 253-254

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the given equation.

$$(x - 2)^2 = \sqrt{4 - 4x + x^2}$$

Solution: $\sqrt{4 - 4x + x^2} = \sqrt{(x - 2)^2} = |x - 2|$, so the equation is $|x - 2|^2 = |x - 2|$. So $|x - 2| = 0$ or $|x - 2| = 1$, and the solutions are

$$x = 1, \quad x = 2, \quad x = 3$$

2. The diagonals of an isosceles trapezoid are perpendicular to each other, and they divide each other into two parts whose ratio is 7 : 17. Find the area of the trapezoid, if its perimeter is $100\sqrt{2}$ units.

Solution: The diagonals of an isosceles trapezoid are equal, and they divide each other in the same ratio, so each diagonal is divided into parts $7x$ and $17x$. The diagonals divide the trapezoid into four right triangles. Two of them are isosceles right triangles, one with legs $17x$ and one with legs $7x$, whose hypotenuses are the parallel sides: $17\sqrt{2}x$ and $7\sqrt{2}x$. The other two have legs $7x$ and $17x$, so the legs of the trapezoid are $\sqrt{49 + 289}x = 13\sqrt{2}x$. The perimeter is

$$(17 + 7 + 2 \cdot 13)\sqrt{2}x = 50\sqrt{2}x = 100\sqrt{2}, \quad x = 2$$

So the diagonals are $24x = 48$ units long, and since they are perpendicular, the area is

$$\frac{48 \cdot 48}{2} = 1152 \text{ square units}$$

3. Solve and discuss the following system of equations, where a is a real parameter.

$$\begin{aligned} ax + 3y &= 12 \\ 12x + ay &= 24 \end{aligned}$$

Solution: Subtracting 3 times the second equation from a times the first one:

$$(a^2 - 36)x = 12a - 72, \quad (a - 6)(a + 6)x = 12(a - 6)$$

If $a \neq \pm 6$: $x = \frac{12}{a + 6}$, and then from the first equation $y = \frac{24}{a + 6}$. This is the unique solution (it satisfies both equations).

If $a = 6$: both equations are equivalent to $2x + y = 4$, so there are infinitely many solutions: $x = t, y = 4 - 2t$, where t is any real number.

If $a = -6$: the equations are $-2x + y = 4$ and $2x - y = 4$, that is, $2x - y = -4$ and $2x - y = 4$, which contradict each other. There is no solution.

4. In the isosceles triangle ABC , $AB = AC = 4$ units. The area of the triangle is 4 square units. Find the angles of the triangle and the length of the side BC .

Solution: The area is $\frac{1}{2} \cdot 4 \cdot 4 \sin \alpha = 4$, so $\sin \alpha = \frac{1}{2}$, and the angle at A is $\alpha = 30^\circ$ or $\alpha = 150^\circ$. The base BC is $2 \cdot 4 \sin \frac{\alpha}{2}$.

If $\alpha = 30^\circ$: the base angles are 75° , and $BC = 8 \sin 15^\circ = 2\sqrt{2}(\sqrt{3} - 1) \approx 2.07$.

If $\alpha = 150^\circ$: the base angles are 15° , and $BC = 8 \sin 75^\circ = 2\sqrt{2}(\sqrt{3} + 1) \approx 7.73$.

(Here $\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$ and $\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$.)

5. Find the value of the parameter a so that one root of the equation

$$x^2 - ax + 2a - 4 = 0$$

is three times the other root.

Solution: The discriminant is $D = a^2 - 4(2a - 4) = (a - 4)^2$, so the roots are

$$x_{1,2} = \frac{a \pm (a - 4)}{2}, \quad x_1 = a - 2, \quad x_2 = 2$$

If $a - 2 = 3 \cdot 2$, then $a = 8$ (the equation $x^2 - 8x + 12 = 0$ has roots 2 and 6). If $2 = 3(a - 2)$, then $a = \frac{8}{3}$ (the roots are $\frac{2}{3}$ and 2). So $a = 8$ or $a = \frac{8}{3}$.

6. One side of a trapezoid is 8 units long, and the sum of the side parallel to it and the height is 12 units. How should we choose the height of the trapezoid so that its area is as large as possible? What is the largest area?

Solution: Let the height be m ; then the other parallel side is $12 - m$, with $0 < m < 12$. The area is

$$t(m) = \frac{8 + (12 - m)}{2} \cdot m = \frac{1}{2}m(20 - m)$$

By the inequality between the arithmetic and geometric means,

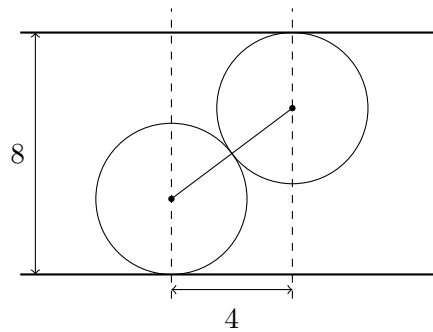
$$\frac{1}{2}m(20 - m) \leq \frac{1}{2} \left(\frac{m + (20 - m)}{2} \right)^2 = 50$$

with equality if $m = 20 - m$, that is, $m = 10$, which is between 0 and 12. So the height should be 10 units (and then the other parallel side is 2 units), and the largest area is 50 square units.

7. Two congruent circles that are tangent to each other are placed between two parallel lines that are 8 units apart. (The circles are also tangent to the lines.) The distance between the lines through the centers of the two circles perpendicular to the given parallel lines is 4 units. Find the radius of the circles.

Solution: Let r be the radius. A circle tangent to both lines would have radius 4, but then the two centers would be on the middle line, 8 units apart, not 4. So one circle touches one line and the other circle touches the other line, and the centers are r away from these lines. The distance of the centers is $2r$ (the circles are tangent); the centers are 4 units apart in the direction of the lines and $8 - 2r$ units apart perpendicular to the lines. By the Pythagorean theorem,

$$(2r)^2 = 4^2 + (8 - 2r)^2, \quad 4r^2 = 80 - 32r + 4r^2, \quad r = \frac{5}{2}$$



8. The orthocenter of the acute triangle ABC is M , and the point of the circumscribed circle opposite to A is A' . Prove that the quadrilateral $BMCA'$ is a parallelogram.

Solution: AA' is a diameter of the circle, so by Thales' theorem, $A'C \perp AC$ and $A'B \perp AB$. The altitude BM is also perpendicular to AC , so $BM \parallel A'C$; and the altitude CM is also perpendicular to AB , so $CM \parallel A'B$. So the opposite sides of the quadrilateral $BMCA'$ are parallel, and it is a parallelogram.

