Problem Set II - 1989

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translated from: KoMaL, October 1989, page 293
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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.
The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.
In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Prove that if the angle opposite the side $BC = a$ of a triangle is 150°, then

   $$b^2 + c^2 = r^2 - 4t\sqrt{3}$$

   where b and c are the other two sides of the triangle, r is the radius of the circumscribed circle, and t is the area of the triangle.

2. Consider the function $f(x) = (x-1)^2 + (x-2)^2 + \cdots + (x-n)^2$, where n is a positive integer. Where does the function take its smallest value, and what is this smallest value?

3. Prove that if $a \geq 0$ and $b \geq 0$, then

   $$\frac{1}{2}(a+b)^2 + \frac{1}{4}(a+b) \geq a\sqrt{b} + b\sqrt{a}$$

   When are the two sides equal?

4. The sides of triangle ABC are $AB = 13$, $BC = 20$, $CA = 21$ units. Find the length of the median to the side BC.

5. Find the equation of the line whose segment between the lines e (with equation $x - y = 2$) and f (with equation $x + 2y = 14$) is divided by the point $P(2,1)$ in the ratio $1:2$, counting from the line e.

6. The base edge of a regular square prism is a, and its height is m (its volume is V_1). The base edge of a regular triangular pyramid is b, and its height is m (its volume is V_2).

   a) Find the ratio $\frac{b}{a}$ if $V_1 = \sqrt{3} \cdot V_2$.

   b) Find the ratio $\frac{m}{a}$ if $V_1 = \sqrt{3} \cdot V_2$ and the lateral surface area of the prism is equal to the lateral surface area of the pyramid (the sum of the areas of its lateral faces).

7. Solve the inequality $\log_3 \frac{1}{x^2} + 3 \cdot \log_x \frac{1}{9} \geq -8$.

8. For which α does the equation $x^2 + \frac{2x}{\sqrt{\cos\alpha}} + \left(\frac{1}{\sin\alpha} + 2\sqrt{2}\right) = 0$ have exactly one real root?

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