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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Prove that if the angle opposite the side $BC = a$ of a triangle is 150° , then

$$b^2 + c^2 = r^2 - 4t\sqrt{3}$$

where b and c are the other two sides of the triangle, r is the radius of the circumscribed circle, and t is the area of the triangle.

Solution: By the Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc \cos 150^\circ = b^2 + c^2 + \sqrt{3}bc$$

The area is $t = \frac{1}{2}bc \sin 150^\circ = \frac{bc}{4}$, so $bc = 4t$. By the Law of Sines, $a = 2r \sin 150^\circ = r$. So

$$b^2 + c^2 = a^2 - \sqrt{3}bc = r^2 - 4t\sqrt{3}$$

2. Consider the function $f(x) = (x-1)^2 + (x-2)^2 + \cdots + (x-n)^2$, where n is a positive integer. Where does the function take its smallest value, and what is this smallest value?

Solution: Expanding, and using $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ and $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$:

$$f(x) = nx^2 - n(n+1)x + \frac{n(n+1)(2n+1)}{6}$$

Completing the square:

$$f(x) = n \left(x - \frac{n+1}{2} \right)^2 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4} = n \left(x - \frac{n+1}{2} \right)^2 + \frac{n(n^2-1)}{12}$$

So the function takes its smallest value at $x = \frac{n+1}{2}$ (the average of $1, 2, \dots, n$), and the smallest value is $\frac{n(n^2-1)}{12}$.

3. Prove that if $a \geq 0$ and $b \geq 0$, then

$$\frac{1}{2}(a+b)^2 + \frac{1}{4}(a+b) \geq a\sqrt{b} + b\sqrt{a}$$

When are the two sides equal?

Solution: The inequality can be written as

$$\frac{a+b}{2} \left(a+b + \frac{1}{2} \right) \geq \sqrt{ab} (\sqrt{a} + \sqrt{b})$$

By the inequality between the arithmetic and geometric means, $\frac{a+b}{2} \geq \sqrt{ab} \geq 0$. Also

$$a+b + \frac{1}{2} - (\sqrt{a} + \sqrt{b}) = \left(\sqrt{a} - \frac{1}{2} \right)^2 + \left(\sqrt{b} - \frac{1}{2} \right)^2 \geq 0$$

so $a+b + \frac{1}{2} \geq \sqrt{a} + \sqrt{b} \geq 0$. Multiplying the two inequalities (all sides are nonnegative) gives the statement.

Equality: if $a = b = 0$, both sides are 0. Otherwise $\frac{a+b}{2} > 0$, and equality needs equality in both inequalities: $a = b$ and $\sqrt{a} = \sqrt{b} = \frac{1}{2}$. So the two sides are equal exactly for $a = b = 0$ and for $a = b = \frac{1}{4}$.

4. The sides of triangle ABC are $AB = 13$, $BC = 20$, $CA = 21$ units. Find the length of the median to the side BC .

Solution: Let A_1 be the midpoint of BC , $s = AA_1$, and $\varphi = \angle AA_1B$; then $\angle AA_1C = 180^\circ - \varphi$. By the Law of Cosines in the triangles AA_1B and AA_1C ,

$$13^2 = s^2 + 10^2 - 20s \cos \varphi, \quad 21^2 = s^2 + 10^2 + 20s \cos \varphi$$

Adding these: $2s^2 + 200 = 169 + 441$, so $s^2 = 205$ and

$$s = \sqrt{205} \approx 14.32$$

5. Find the equation of the line whose segment between the lines e (with equation $x - y = 2$) and f (with equation $x + 2y = 14$) is divided by the point $P(2, 1)$ in the ratio 1 : 2, counting from the line e .

Solution: Let the endpoints of the segment be $E(t, t-2)$ on e and F on f . Since $PF = 2PE$ and P is between them, $\overrightarrow{PF} = -2\overrightarrow{PE}$, so

$$F = P - 2(E - P) = 3P - 2E = (6 - 2t, 7 - 2t)$$

F is on f : $(6 - 2t) + 2(7 - 2t) = 14$, so $t = 1$. So $E(1, -1)$ and $F(4, 5)$, and the line through E and P has slope 2. Its equation is

$$2x - y = 3$$

6. The base edge of a regular square prism is a , and its height is m (its volume is V_1). The base edge of a regular triangular pyramid is b , and its height is m (its volume is V_2).

a) Find the ratio $\frac{b}{a}$ if $V_1 = \sqrt{3} \cdot V_2$.

b) Find the ratio $\frac{m}{a}$ if $V_1 = \sqrt{3} \cdot V_2$ and the lateral surface area of the prism is equal to the lateral surface area of the pyramid (the sum of the areas of its lateral faces).

Solution: a) $V_1 = a^2m$ and $V_2 = \frac{1}{3} \cdot \frac{\sqrt{3}}{4}b^2 \cdot m$, so $V_1 = \sqrt{3}V_2$ means

$$a^2m = \frac{b^2m}{4}, \quad \left(\frac{b}{a}\right)^2 = 4, \quad \frac{b}{a} = 2$$

b) The apex of the pyramid is above the center of the base triangle, which is $\frac{1}{3} \cdot \frac{\sqrt{3}}{2}b = \frac{b}{2\sqrt{3}}$ away from each base edge. So the height m_1 of a lateral face (to the side b) satisfies

$$m_1^2 = m^2 + \frac{b^2}{12} = m^2 + \frac{a^2}{3}$$

using $b = 2a$ from part a). The lateral surface areas are equal:

$$4am = 3 \cdot \frac{2a \cdot m_1}{2}, \quad 4m = 3\sqrt{m^2 + \frac{a^2}{3}}, \quad 16m^2 = 9m^2 + 3a^2$$

So $7m^2 = 3a^2$, and

$$\frac{m}{a} = \sqrt{\frac{3}{7}} \approx 0.65$$

7. Solve the inequality $\log_3 \frac{1}{x^2} + 3 \cdot \log_x \frac{1}{9} \geq -8$.

Solution: We need $x > 0$ and $x \neq 1$. Let $z = \log_3 x \neq 0$; then $\log_3 \frac{1}{x^2} = -2z$ and $\log_x \frac{1}{9} = -2 \log_x 3 = -\frac{2}{z}$. So

$$-2z - \frac{6}{z} \geq -8, \quad z + \frac{3}{z} - 4 \leq 0, \quad \frac{(z-1)(z-3)}{z} \leq 0$$

A sign chart gives $z < 0$ or $1 \leq z \leq 3$. Since $\log_3$ is increasing, the solution is

$$0 < x < 1 \quad \text{or} \quad 3 \leq x \leq 27$$

8. For which α does the equation $x^2 + \frac{2x}{\sqrt{\cos \alpha}} + \left(\frac{1}{\sin \alpha} + 2\sqrt{2} \right) = 0$ have exactly one real root?

Solution: The equation is defined if $\cos \alpha > 0$ and $\sin \alpha \neq 0$. It has exactly one real root if its discriminant is zero:

$$\frac{4}{\cos \alpha} - 4 \left(\frac{1}{\sin \alpha} + 2\sqrt{2} \right) = 0, \quad \frac{\sin \alpha - \cos \alpha}{\sin \alpha \cos \alpha} = 2\sqrt{2}, \quad \frac{\sin \alpha - \cos \alpha}{\sqrt{2}} = \sin 2\alpha$$

The left-hand side is $\sin \left(\alpha - \frac{\pi}{4} \right)$, so $\sin \left(\alpha - \frac{\pi}{4} \right) = \sin 2\alpha$, and

$$2\alpha = \alpha - \frac{\pi}{4} + 2n\pi \quad \text{or} \quad 2\alpha = \pi - \left(\alpha - \frac{\pi}{4} \right) + 2n\pi$$

that is, $\alpha = -\frac{\pi}{4} + 2n\pi$ or $\alpha = \frac{5\pi}{12} + \frac{2n\pi}{3}$. Now check the condition $\cos \alpha > 0$ (and $\sin \alpha \neq 0$, which holds for all of these). $\alpha = -\frac{\pi}{4} + 2n\pi$ is good. In the second family, $\frac{5\pi}{12} + 2k\pi$ is good, $\frac{13\pi}{12} + 2k\pi$ is not ($\cos \alpha < 0$), and $\frac{21\pi}{12} + 2k\pi = -\frac{\pi}{4} + 2(k+1)\pi$ is in the first family. So the equation has exactly one real root for

$$\alpha = -\frac{\pi}{4} + 2k\pi \quad \text{or} \quad \alpha = \frac{5\pi}{12} + 2k\pi, \quad k \text{ is an integer}$$

(and then the root is $x = -\frac{1}{\sqrt{\cos \alpha}}$).