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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. The parallel sides of trapezoid $ABCD$ are $AB = 4$ units and $DC = 1$ unit long, and its legs are $AD = 2$ units and $BC = 3$ units long. Find the area of the trapezoid.

Solution: Let m be the height, and let the projections of the legs AD and BC onto AB be p and q . Then $p + q = 4 - 1 = 3$, and

$$p^2 + m^2 = 4, \quad q^2 + m^2 = 9$$

Subtracting: $q^2 - p^2 = (q - p) \cdot 3 = 5$, so $q - p = \frac{5}{3}$, $p = \frac{2}{3}$, and $m^2 = 4 - \frac{4}{9} = \frac{32}{9}$, $m = \frac{4\sqrt{2}}{3}$. The area is

$$\frac{4+1}{2} \cdot \frac{4\sqrt{2}}{3} = \frac{10\sqrt{2}}{3} \approx 4.71$$

2. In triangle ABC , $AC = 6$ units and $BC = 4$ units. The angle opposite the side AC is twice the angle opposite the side BC . Find the exact length of the side AB .

Solution: Let α be the angle at A (opposite BC); then the angle at B (opposite AC) is 2α . By the Law of Sines,

$$\frac{AC}{BC} = \frac{\sin 2\alpha}{\sin \alpha} = 2 \cos \alpha = \frac{6}{4}, \quad \cos \alpha = \frac{3}{4}$$

The angle at C is $180^\circ - 3\alpha$, so again by the Law of Sines,

$$\frac{AB}{BC} = \frac{\sin 3\alpha}{\sin \alpha} = 3 - 4 \sin^2 \alpha = 3 - 4 \cdot \frac{7}{16} = \frac{5}{4}$$

So $AB = 5$ units.

(Without trigonometry: extend CB beyond B to the point D with $BD = BA$. Then the triangle ABD is isosceles, and its exterior angle at B is 2α , so $\angle BDA = \alpha$. So the triangles ABC and DAC are similar (they have the angle α and the common angle at C), and $\frac{DC}{AC} = \frac{AC}{BC}$, that is, $\frac{AB+4}{6} = \frac{6}{4}$, $AB = 5$.)

3. For what values of the real parameter m are the roots of the equation

$$2x^2 - (5m + 10)x + 2(m^2 + m - 6) = 0$$

in the interval $[-4, -3]$? In this case, is there an m for which the two roots of the equation are equal?

Solution: The discriminant is

$$D = (5m + 10)^2 - 16(m^2 + m - 6) = 9m^2 + 84m + 196 = (3m + 14)^2$$

so the roots are

$$x_{1,2} = \frac{5m + 10 \pm (3m + 14)}{4}, \quad x_1 = 2m + 6, \quad x_2 = \frac{m}{2} - 1$$

$-4 \leq 2m + 6 \leq -3$ gives $-5 \leq m \leq -\frac{9}{2}$, and $-4 \leq \frac{m}{2} - 1 \leq -3$ gives $-6 \leq m \leq -4$. Both roots are in the interval $[-4, -3]$ if

$$-5 \leq m \leq -\frac{9}{2}$$

The roots are equal if $D = 0$, that is, $m = -\frac{14}{3}$, which is in this interval: then $x_1 = x_2 = -\frac{10}{3}$.

4. Find the largest subset of the real numbers on which the following expressions are defined.

$$\text{a) } \log(1 - 2\sin 2x) \qquad \text{b) } \log(1 + \tan 2x)$$

Solution: a) We need $1 - 2\sin 2x > 0$, that is, $\sin 2x < \frac{1}{2}$. On the unit circle, this means $\frac{5\pi}{6} + 2k\pi < 2x < \frac{13\pi}{6} + 2k\pi$, so the expression is defined for

$$\frac{5\pi}{12} + k\pi < x < \frac{13\pi}{12} + k\pi, \quad k \text{ is an integer}$$

b) We need $\tan 2x > -1$ (this includes that $\tan 2x$ is defined). Since $\tan$ is increasing between any two neighboring vertical asymptotes, this means

$$-\frac{\pi}{4} + n\pi < 2x < \frac{\pi}{2} + n\pi$$

so the expression is defined for

$$-\frac{\pi}{8} + \frac{n\pi}{2} < x < \frac{\pi}{4} + \frac{n\pi}{2}, \quad n \text{ is an integer}$$

5. A normal vector of the line of side AB of rectangle $ABCD$ is $\mathbf{n}(1, -3)$, and the vertex A is $A(2, 1)$. The vertex C (opposite to A) is on the line $x - y = 1$. Find the coordinates of the missing vertices of the rectangle, if the diagonals are $5\sqrt{2}$ units long.

Solution: The line AB is $(x - 2) - 3(y - 1) = 0$, that is, $x - 3y = -1$. The vertex C is on the circle with center A and radius $5\sqrt{2}$, $(x - 2)^2 + (y - 1)^2 = 50$, and on the line $y = x - 1$:

$$2(x - 2)^2 = 50, \quad x - 2 = \pm 5$$

so $C_1(7, 6)$ or $C_2(-3, -4)$. The vertex B is the foot of the perpendicular from C to the line AB , and $D = A + C - B$ (the diagonals bisect each other).

For $C_1(7, 6)$: the perpendicular is $3x + y = 27$; with $x = 3y - 1$ we get $B_1(8, 3)$, and $D_1(1, 4)$.

For $C_2(-3, -4)$: the perpendicular is $3x + y = -13$; with $x = 3y - 1$ we get $B_2(-4, -1)$, and $D_2(3, -2)$.

There are two such rectangles: $B(8, 3)$, $C(7, 6)$, $D(1, 4)$, or $B(-4, -1)$, $C(-3, -4)$, $D(3, -2)$.

6. Solve the given system of equations.

$$\begin{aligned}x^{\log_4 y} + y^{\log_4 x} &= 4 \\ \log_2 x - \log_2 y &= 1\end{aligned}$$

Solution: We need $x > 0$ and $y > 0$. Since $x^{\log_4 y} = 4^{\log_4 x \cdot \log_4 y} = y^{\log_4 x}$, the first equation says $2x^{\log_4 y} = 4$, that is, $x^{\log_4 y} = 2$. Taking $\log_2$:

$$\log_4 y \cdot \log_2 x = 1, \quad \log_2 x \cdot \log_2 y = 2$$

Let $u = \log_2 x$ and $v = \log_2 y$. Then $uv = 2$ and $u - v = 1$, so $u(u - 1) = 2$, $u^2 - u - 2 = 0$, and $u = 2$ or $u = -1$.

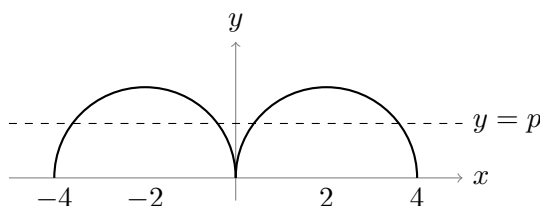
If $u = 2$: $v = 1$, so $x = 4$, $y = 2$. If $u = -1$: $v = -2$, so $x = \frac{1}{2}$, $y = \frac{1}{4}$.

7. Depending on the value of the real number p , how many real roots does the equation

$$\sqrt{4|x| - x^2} = p$$

have?

Solution: For $x \geq 0$, the graph of $y = \sqrt{4x - x^2} = \sqrt{4 - (x - 2)^2}$ is the upper half of the circle with center $(2, 0)$ and radius 2; the function is even, so for $x \leq 0$ its graph is the upper half of the circle with center $(-2, 0)$ and radius 2. The number of roots is the number of intersection points of this graph with the horizontal line $y = p$:



- If $p < 0$ or $p > 2$: no roots.
- If $p = 0$: 3 roots, $x = 0$ and $x = \pm 4$.
- If $0 < p < 2$: 4 roots, $x = \pm \left(2 + \sqrt{4 - p^2}\right)$ and $x = \pm \left(2 - \sqrt{4 - p^2}\right)$.
- If $p = 2$: 2 roots, $x = \pm 2$.

8. Find the pairs of integers x, y for which $xy + 3x - y = 20$.

Solution: The equation can be written as

$$xy + 3x - y - 3 = 17, \quad (x - 1)(y + 3) = 17$$

Since 17 is a prime number, $x - 1$ is 1, 17, -1, or -17, and $y + 3 = \frac{17}{x - 1}$. The solutions are

$$(x, y) = (2, 14), (18, -2), (0, -20), (-16, -4)$$