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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Put the following numbers in increasing order.

$$\left(\log_2 \left(\frac{1}{2}\right)^{-2}\right)^{\sin 750^\circ}, \quad \log(\cos 0^\circ + \cos 36^\circ + \cdots + \cos 144^\circ), \quad \cos^{16} 1990^\circ$$

Solution: $\sin 750^\circ = \sin 30^\circ = \frac{1}{2}$ and $\log_2 \left(\frac{1}{2}\right)^{-2} = \log_2 4 = 2$, so the first number is $2^{\frac{1}{2}} = \sqrt{2}$.

The sum is $\cos 0^\circ + \cos 36^\circ + \cos 72^\circ + \cos 108^\circ + \cos 144^\circ$, and since $\cos(180^\circ - \varphi) = -\cos \varphi$, $\cos 36^\circ + \cos 144^\circ = 0$ and $\cos 72^\circ + \cos 108^\circ = 0$. So the sum is 1, and the second number is $\log 1 = 0$.

$1990^\circ = 5 \cdot 360^\circ + 190^\circ$, so $\cos 1990^\circ = \cos 190^\circ = -\cos 10^\circ$, and the third number is $\cos^{16} 10^\circ$, which is between 0 and 1. So the increasing order is

$$\log(\cos 0^\circ + \cdots + \cos 144^\circ) < \cos^{16} 1990^\circ < \left(\log_2 \left(\frac{1}{2}\right)^{-2}\right)^{\sin 750^\circ}$$

2. Let F be the midpoint of the side AB of a square, let H be the point dividing the diagonal from A into thirds that is closer to A , and let N be the point dividing the diagonal from B into fourths that is closest to B . What percent of the area of the square is the area of the triangle NFH ?

Solution: Let the square be $A(0, 0)$, $B(1, 0)$, $C(1, 1)$, $D(0, 1)$. Then $F\left(\frac{1}{2}, 0\right)$, $H\left(\frac{1}{3}, \frac{1}{3}\right)$ (on AC), and $N = B + \frac{1}{4}(D - B) = \left(\frac{3}{4}, \frac{1}{4}\right)$ (on BD). The area of the triangle is

$$\frac{1}{2} |\overrightarrow{FH} \times \overrightarrow{FN}| = \frac{1}{2} \left| \left(-\frac{1}{6}\right) \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{4} \right| = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$$

So the area of the triangle is $\frac{1}{16} = 6.25\%$ of the area of the square.

3. Of three consecutive terms of a geometric sequence, the third one is -2 . If we put it in front of the first two, we get three consecutive terms of an arithmetic sequence. What were the original three numbers?

Solution: Let the ratio be q ; the three terms are $-\frac{2}{q^2}$, $-\frac{2}{q}$, -2 . The numbers -2 , $-\frac{2}{q^2}$, $-\frac{2}{q}$ form an arithmetic sequence, so the middle one is the average of the other two:

$$-\frac{4}{q^2} = -2 - \frac{2}{q}, \quad q^2 + q - 2 = 0, \quad (q-1)(q+2) = 0$$

If $q = 1$: the numbers are $-2, -2, -2$ (and $-2, -2, -2$ is also arithmetic). If $q = -2$: the numbers are $-\frac{1}{2}, 1, -2$, and indeed $-2, -\frac{1}{2}, 1$ is an arithmetic sequence with difference $\frac{3}{2}$.

4. Solve the given equation over the real numbers.

$$(\log_2(x^2 - 4x + 3))^2 - (3 + \log_2 15) \log_2(x^2 - 4x + 3) + 3 \log_2 15 = 0$$

Solution: We need $x^2 - 4x + 3 = (x-1)(x-3) > 0$, that is, $x < 1$ or $x > 3$. With $t = \log_2(x^2 - 4x + 3)$, the equation is

$$t^2 - (3 + \log_2 15)t + 3 \log_2 15 = 0, \quad (t-3)(t - \log_2 15) = 0$$

If $t = 3$: $x^2 - 4x + 3 = 8$, $x^2 - 4x - 5 = 0$, so $x = -1$ or $x = 5$. If $t = \log_2 15$: $x^2 - 4x + 3 = 15$, $x^2 - 4x - 12 = 0$, so $x = -2$ or $x = 6$. All four are in the domain, so the solutions are

$$x = -2, \quad x = -1, \quad x = 5, \quad x = 6$$

5. Is there a pair of real numbers that satisfies the equation

$$x^8 + x^6 + x^2 + 1 = 4x^4(1 - 2^y)?$$

Solution: No. If $x = 0$, the left-hand side is 1 and the right-hand side is 0. If $x \neq 0$, divide by $x^4 > 0$:

$$\left(x^4 + \frac{1}{x^4}\right) + \left(x^2 + \frac{1}{x^2}\right) = 4(1 - 2^y)$$

The sum of a positive number and its reciprocal is at least 2, so the left-hand side is at least 4. But $2^y > 0$, so the right-hand side is less than 4. So there is no such pair.

6. Find the intersection points of the circle with center $K(-2, 1)$ and radius $r = 5$ units, and the parabola with focus $F\left(1, -\frac{11}{4}\right)$ and directrix $y = -\frac{13}{4}$.

Solution: The circle is $(x+2)^2 + (y-1)^2 = 25$. The vertex of the parabola is halfway between the focus and the directrix, at $(1, -3)$, and the distance of the focus from the vertex is $\frac{1}{4}$, so the parabola is

$$(x-1)^2 = 4 \cdot \frac{1}{4}(y+3), \quad y = (x-1)^2 - 3 = x^2 - 2x - 2$$

Substituting into the equation of the circle, $(x+2)^2 + (x^2 - 2x - 3)^2 = 25$, which simplifies to

$$x^4 - 4x^3 - x^2 + 16x - 12 = 0$$

Integer roots must divide 12; trying them, the polynomial factors as $(x-1)(x-2)(x-3)(x+2) = 0$. The intersection points are

$$(1, -3), \quad (2, -2), \quad (3, 1), \quad (-2, 6)$$

7. The diagonals of the convex kite $ABCD$ intersect at M . Let $P_1, P_2, P_3, P_4, P_5, P_6$ be points of the segments AM, MB, BC, CM, MD, DA , in this order, such that the pairs of lines P_1P_2 and AB , P_2P_3 and MC , P_3P_4 and BM , P_4P_5 and CD , P_5P_6 and MA are parallel. Prove that P_6P_1 is also parallel to DM .

Solution: We use the intercept theorem (a line parallel to one side of a triangle divides the other two sides in the same ratio) five times:

$$\begin{aligned} P_1P_2 \parallel AB \text{ in triangle } MAB : \quad & \frac{AP_1}{P_1M} = \frac{BP_2}{P_2M} \\ P_2P_3 \parallel MC \text{ in triangle } BMC : \quad & \frac{BP_2}{P_2M} = \frac{BP_3}{P_3C} \\ P_3P_4 \parallel BM \text{ in triangle } CBM : \quad & \frac{BP_3}{P_3C} = \frac{MP_4}{P_4C} \\ P_4P_5 \parallel CD \text{ in triangle } MCD : \quad & \frac{MP_4}{P_4C} = \frac{MP_5}{P_5D} \\ P_5P_6 \parallel MA \text{ in triangle } DMA : \quad & \frac{MP_5}{P_5D} = \frac{AP_6}{P_6D} \end{aligned}$$

So $\frac{AP_1}{P_1M} = \frac{AP_6}{P_6D}$: in triangle AMD , the points P_1 and P_6 divide the sides AM and AD in the same ratio. By the converse of the intercept theorem, $P_6P_1 \parallel MD$.

(The proof works for every convex quadrilateral, not only for kites.)

8. Let α, β, γ be the angles of a triangle in radians. Prove that

$$27(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) \leq 8\pi^3$$

Solution: By the inequality between the arithmetic and geometric means of the three positive numbers $\alpha + \beta, \beta + \gamma, \gamma + \alpha$:

$$(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) \leq \left(\frac{2(\alpha + \beta + \gamma)}{3} \right)^3 = \left(\frac{2\pi}{3} \right)^3 = \frac{8\pi^3}{27}$$

Multiplying by 27 gives the statement. Equality holds for the equilateral triangle.