

# Problem Set IV - 1989 (February 1989)

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the following expression.

$$F(x, y) = \frac{x^3 - x^2y + xy^2 - x - y^3 + y}{x^3 - x^2y + xy^2 + x - y^3 - y}$$

a) Simplify  $F(x, y)$ .

b) Show that if  $x = \frac{k(1 - z^2)}{1 + z^2}$  and  $y = \frac{2kz}{1 + z^2}$ , then  $F(x, y)$  does not depend on  $z$ .

2. For what value of the real parameter  $m$  is the following system of equations solvable?

$$\begin{aligned} 3x + y &= 2 \\ x + 2y &= m + 8 \\ (m - 4)x + y &= -10 \end{aligned}$$

3. A container in the shape of a right circular cone with radius  $r$  and height  $m$  is filled with liquid up to half of its height. We turn the cone upside down so that its apex points down. How high will the liquid be in the container, measured from the apex?

4. Solve the following equations over the real numbers.

$$\text{a) } x^{\log_x(x-2)^2} = 9 \qquad \text{b) } x^{\log_{x^2}(x^2-5)} = 2$$

5. Let  $a, b, c$  be consecutive terms of a geometric sequence. Prove that

$$\frac{1}{3} \leq \frac{ax^2 + bx + c}{ax^2 - bx + c} \leq 3$$

6. Solve the quadratic equation whose roots satisfy the following conditions.

$$4x_1x_2 - 5(x_1 + x_2) = -4, \quad \text{and} \quad (x_1 - 1)(x_2 - 1) = \frac{1}{1 + p}$$

Find the value of  $p$  for which  $x_1^2 + x_2^2 = 11$ .

7. Solve the given equation.

$$3 \cos^2 \alpha + \cos 2\alpha - 2 \cos \alpha - \sin 2\alpha - \sin \alpha = 0$$

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8. On one leg of a right angle, we mark the fixed points  $A$  and  $B$ , and on the other leg the variable point  $L$ . The projections of the vertex  $O$  of the right angle onto the lines  $LA$  and  $LB$  are  $P$  and  $Q$ .
- a) Prove that the line  $PQ$  intersects the leg containing  $A$  and  $B$  at a fixed point  $M$ .
  - b) Find the ratio in which the point  $M$  divides the segment  $AB$ .