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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the following expression.

$$F(x, y) = \frac{x^3 - x^2y + xy^2 - x - y^3 + y}{x^3 - x^2y + xy^2 + x - y^3 - y}$$

a) Simplify $F(x, y)$.

b) Show that if $x = \frac{k(1 - z^2)}{1 + z^2}$ and $y = \frac{2kz}{1 + z^2}$, then $F(x, y)$ does not depend on z .

Solution: a) Grouping the terms,

$$x^3 - x^2y + xy^2 - y^3 = x^2(x - y) + y^2(x - y) = (x - y)(x^2 + y^2)$$

so the numerator is $(x - y)(x^2 + y^2 - 1)$, and the denominator is $(x - y)(x^2 + y^2 + 1)$. So for $x \neq y$,

$$F(x, y) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

b) Since $(1 - z^2)^2 + (2z)^2 = (1 + z^2)^2$,

$$x^2 + y^2 = k^2 \cdot \frac{(1 - z^2)^2 + 4z^2}{(1 + z^2)^2} = k^2$$

so $F(x, y) = \frac{k^2 - 1}{k^2 + 1}$, which does not depend on z (for the values of z with $x \neq y$).

Note: In KoMaL, this problem says $y = \frac{2 + z}{1 + z^2}$. That is most likely a misprint for $\frac{2kz}{1 + z^2}$: with $2 + z$, the value of F does depend on z .

2. For what value of the real parameter m is the following system of equations solvable?

$$\begin{aligned} 3x + y &= 2 \\ x + 2y &= m + 8 \\ (m - 4)x + y &= -10 \end{aligned}$$

Solution: The first two equations have a unique solution for every m : from the first one $y = 2 - 3x$, and then $x + 4 - 6x = m + 8$, so

$$x = -\frac{m + 4}{5}, \quad y = \frac{3m + 22}{5}$$

The system is solvable if and only if this pair also satisfies the third equation:

$$-\frac{(m - 4)(m + 4)}{5} + \frac{3m + 22}{5} = -10, \quad m^2 - 3m - 88 = 0, \quad (m - 11)(m + 8) = 0$$

For $m = 11$ the solution is $x = -3, y = 11$; for $m = -8$ it is $x = \frac{4}{5}, y = -\frac{2}{5}$.

3. A container in the shape of a right circular cone with radius r and height m is filled with liquid up to half of its height. We turn the cone upside down so that its apex points down. How high will the liquid be in the container, measured from the apex?

Solution: At first the cone stands on its base, and the liquid fills it up to height $\frac{m}{2}$. The empty part is the top of the cone, a cone similar to the whole one with ratio $\frac{1}{2}$, so its volume is $\frac{1}{8}V$, where V is the volume of the container. So the volume of the liquid is $\frac{7}{8}V$.

When the apex points down, the liquid fills a cone similar to the container, with some height h . Its volume is $\left(\frac{h}{m}\right)^3 V = \frac{7}{8}V$, so

$$h = m \sqrt[3]{\frac{7}{8}} = \frac{\sqrt[3]{7}}{2} m \approx 0.956m$$

(The answer does not depend on r .)

4. Solve the following equations over the real numbers.

$$\text{a) } x^{\log_x(x-2)^2} = 9 \quad \text{b) } x^{\log_{x^2}(x^2-5)} = 2$$

Solution: a) We need $x > 0$, $x \neq 1$, and $(x - 2)^2 > 0$, that is, $x \neq 2$. Then the left-hand side is $(x - 2)^2$, so $(x - 2)^2 = 9$, $x = 5$ or $x = -1$. Only $x = 5$ is allowed.

b) The power with a non-integer exponent needs a positive base, so $x > 0$; also $x^2 \neq 1$ and $x^2 - 5 > 0$, so $x > \sqrt{5}$.

Since $\log_{x^2}(x^2 - 5) = \frac{\log_x(x^2 - 5)}{\log_x x^2} = \frac{1}{2} \log_x(x^2 - 5)$,

$$x^{\log_{x^2}(x^2-5)} = \left(x^{\log_x(x^2-5)}\right)^{\frac{1}{2}} = \sqrt{x^2 - 5}$$

So $\sqrt{x^2 - 5} = 2$, $x^2 = 9$, and $x = 3$.

Note: In KoMaL, the right-hand side of a) is printed as g , which is most likely a misprint for 9.

5. Let a, b, c be consecutive terms of a geometric sequence. Prove that

$$\frac{1}{3} \leq \frac{ax^2 + bx + c}{ax^2 - bx + c} \leq 3$$

Solution: Let the ratio be q ; then $a \neq 0, q \neq 0, b = aq, c = aq^2$, and

$$\frac{ax^2 + bx + c}{ax^2 - bx + c} = \frac{x^2 + qx + q^2}{x^2 - qx + q^2}$$

The denominator is $\left(x - \frac{q}{2}\right)^2 + \frac{3q^2}{4} > 0$, so the expression is defined for every x , and we can multiply by the denominator. The two inequalities are equivalent to

$$\begin{aligned} x^2 - qx + q^2 &\leq 3(x^2 + qx + q^2) \Leftrightarrow 0 \leq 2(x + q)^2 \\ x^2 + qx + q^2 &\leq 3(x^2 - qx + q^2) \Leftrightarrow 0 \leq 2(x - q)^2 \end{aligned}$$

which are both true. (Equality holds on the left for $x = -q$ and on the right for $x = q$.)

6. Solve the quadratic equation whose roots satisfy the following conditions.

$$4x_1x_2 - 5(x_1 + x_2) = -4, \quad \text{and} \quad (x_1 - 1)(x_2 - 1) = \frac{1}{1+p}$$

Find the value of p for which $x_1^2 + x_2^2 = 11$.

Solution: Let $s = x_1 + x_2$ and $t = x_1x_2$. The conditions say $4t - 5s = -4$ and $t - s + 1 = \frac{1}{1+p}$. From the first one, $t = \frac{5s - 4}{4}$, and then the second one gives $\frac{s}{4} = \frac{1}{1+p}$. So

$$s = \frac{4}{1+p}, \quad t = \frac{5}{1+p} - 1 = \frac{4-p}{1+p}$$

and the equation is $x^2 - sx + t = 0$, that is,

$$(1+p)x^2 - 4x + 4 - p = 0, \quad p \neq -1$$

Now let $u = \frac{1}{1+p}$; then $s = 4u$ and $t = 5u - 1$, so

$$x_1^2 + x_2^2 = s^2 - 2t = 16u^2 - 10u + 2 = 11, \quad 16u^2 - 10u - 9 = 0, \quad u = \frac{9}{8} \text{ or } u = -\frac{1}{2}$$

$u = \frac{9}{8}$ gives $p = -\frac{1}{9}$ and the equation $8x^2 - 36x + 37 = 0$; $u = -\frac{1}{2}$ gives $p = -3$ and the equation $2x^2 + 4x - 7 = 0$.

Both equations have real roots (their discriminants are 112 and 72), so both values work: $p = -\frac{1}{9}$ or $p = -3$.

7. Solve the given equation.

$$3 \cos^2 \alpha + \cos 2\alpha - 2 \cos \alpha - \sin 2\alpha - \sin \alpha = 0$$

Solution: Write $s = \sin \alpha$ and $c = \cos \alpha$. Using $\cos 2\alpha = c^2 - s^2$ and $\sin 2\alpha = 2sc$, the equation is

$$4c^2 - s^2 - 2c - 2sc - s = 0$$

Using $c^2 = 1 - s^2 = (1 - s)(1 + s)$, the left-hand side becomes

$$4(1 - s)(1 + s) - 2c(1 + s) - s(1 + s) = (1 + s)(4 - 5s - 2c)$$

If $\sin \alpha = -1$: $\alpha = -\frac{\pi}{2} + 2k\pi$.

If $5 \sin \alpha + 2 \cos \alpha = 4$: with $t = \tan \frac{\alpha}{2}$ (note that $\alpha = \pi + 2k\pi$ is not a solution), $\sin \alpha = \frac{2t}{1+t^2}$ and $\cos \alpha = \frac{1-t^2}{1+t^2}$, so

$$10t + 2 - 2t^2 = 4 + 4t^2, \quad 3t^2 - 5t + 1 = 0, \quad t = \frac{5 \pm \sqrt{13}}{6}$$

The solutions are

$$\alpha = -\frac{\pi}{2} + 2k\pi$$

$$\alpha = 2 \tan^{-1} \frac{5 + \sqrt{13}}{6} + 2k\pi \approx 110.23^\circ + k \cdot 360^\circ$$

$$\alpha = 2 \tan^{-1} \frac{5 - \sqrt{13}}{6} + 2k\pi \approx 26.17^\circ + k \cdot 360^\circ$$

8. On one leg of a right angle, we mark the fixed points A and B , and on the other leg the variable point L . The projections of the vertex O of the right angle onto the lines LA and LB are P and Q .

a) Prove that the line PQ intersects the leg containing A and B at a fixed point M .

b) Find the ratio in which the point M divides the segment AB .

Solution: Use coordinates: $O(0, 0)$, $A(a, 0)$, $B(b, 0)$ with $0 < a < b$, and $L(0, l)$ with $l > 0$. The projection of O onto the line LA (whose direction is $(a, -l)$) is

$$P = \left(\frac{al^2}{a^2 + l^2}, \frac{a^2l}{a^2 + l^2} \right), \quad \text{and similarly} \quad Q = \left(\frac{bl^2}{b^2 + l^2}, \frac{b^2l}{b^2 + l^2} \right)$$

(One can check that P is on the line $\frac{x}{a} + \frac{y}{l} = 1$ and $\overrightarrow{OP} \perp (a, -l)$.)

a) Let $M = (m, 0)$ with $m = \frac{ab}{a+b}$. The points M, P, Q are collinear if the determinant of $\overrightarrow{MP}$ and $\overrightarrow{MQ}$ is zero.

Multiplied by $\frac{(a^2 + l^2)(b^2 + l^2)}{l}$, this determinant is

$$\begin{aligned} & (al^2 - m(a^2 + l^2))b^2 - (bl^2 - m(b^2 + l^2))a^2 \\ &= abl^2(b - a) - ml^2(b^2 - a^2) = l^2(b - a)(ab - m(a + b)) = 0 \end{aligned}$$

So the line PQ always goes through the point M , which does not depend on L . Its distance from O satisfies $\frac{1}{OM} = \frac{1}{OA} + \frac{1}{OB}$.

b) $OM = \frac{ab}{a+b} < a$, so M is outside the segment AB , on the side of A : M divides AB externally. Since

$$MA = a - \frac{ab}{a+b} = \frac{a^2}{a+b}, \quad MB = b - \frac{ab}{a+b} = \frac{b^2}{a+b}$$

the ratio is

$$MA : MB = OA^2 : OB^2$$